THE TURÁN EXPRESSIONS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "THE TURÁN EXPRESSIONS", submitted by M.N. BAJAJ in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The aim of this thesis is to survey the literature relating to the Turán expression. Most of the results concerning Turán expressions and their generalizations have been stated in brief. Several methods have been discussed in detail for proving the Turán inequality for classical orthogonal polynomials and Bessel functions.

Turán expressions for the Hermite, the general Laguerre and the Ultraspherical polynomials satisfy differential equations, each of the third order, and these have been derived. In the end miscellaneous results are collected.

Finally, a reasonably complete bibliography of the literature on the Turán expressions is included.

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TABLE OF CONTENTS

INTRODUCTION

During the investigation of the zeros of the Legendre polynomials, P. Turán observed the following inequality for the Legendre polynomials:

$$\Delta_n(P) = P_n^2(x) - P_{n+1}(x)P_{n-1}(x) \geq 0$$

for $n \geq 1$, $|x| \leq 1$ with equality for $|x| = 1$, where $P_n(x)$ denotes the Legendre polynomial of the nth degree.

As far back as 1870, Lommel [58] studied such inequalities for the cylindrical functions. Later on, Demir [26] proved the above inequality for the Hermite polynomials.

In a written communication to G. Szegö, Turán indicated the proof of the above inequality. Szegö then gave four different proofs of this inequality for the Legendre polynomials and in his third proof he also proved the analogous inequalities for the Ultraspherical, the general Laguerre and the Hermite polynomials.

In recent years an extensive literature has developed around this subject. In particular the above inequality has been extended to the case of some orthogonal polynomials and other functions such as the Bessel functions etc.

In the first chapter, we review a few results of the Turán

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expression and its generalizations for certain classical orthogonal polynomials. We also state the generating functions for the generalized Turán expressions for the Hermite polynomials, a q-analog. of the Hermite polynomials and the Bessel functions.

The purpose of the second chapter is to exhibit different types of methods used to prove the Turán inequality for the orthogonal polynomials. Four methods, as proved by G. Szegö for the Legendre polynomials, have been discussed in detail. With the help of the fourth method, we have proved the Turán inequality for the Hermite, general Laguerre and the Ultaspherical polynomials. In the end of this chapter, Szász's method has been discussed for getting the Turán inequality for the general Laguerre, the Ultraspherical and the Bessel function.

Turán expressions for the classical orthogonal polynomials satisfy differential equations, each of the third order. These have been obtained in the third chapter.

In the fourth chapter, miscellaneous results are collected. The Turán inequality for certain types of polynomials which can be represented symbolically is satisfied under certain sufficient conditions. Under restricted conditions on α and x the Jacobi polynomial with $\beta = -\alpha$ satisfies Turán inequality. Moreover, it has been shown that the monic Jacobi polynomials satisfy an inequality of the Turán type.

CHAPTER I

GENERAL REMARKS

As remarked in the introduction P. Turán observed the inequality while studying the properties of the zeros of the Legendre polynomials. Szegö [50] gave four different proofs of this remarkable inequality for the Legendre polynomials.

For convenience, we shall define the Turán expression for a sequence of functions $\{f_n(x)\}$ to be

$$\begin{aligned} \Delta_n(f) &= f_n^2(x) - f_{n+1}(x)f_{n-1}(x) \qquad && (n \geq 1) \\ &= 1 && (n = 0) \end{aligned}$$

Several authors have applied different techniques for proving the Turán inequality for the classical orthogonal polynomials. We shall take the results one by one.

For the Hermite polynomials the following results have been proved:

Demir [26] proved

$$\Delta_{n+1}(H) = n! \sum_{p=o}^{n} \frac{H_p^2(x)}{p!}$$

Al-Salam [2] proved

$$\Delta_n(H^{(k)}) \geq (n-1)!(k!)k \quad ,$$

where $H^{(k)}$ is the kth derivative.

Al-Salam [1] proved that

$$D_n(H) = \Delta_n(H) + n(n-1)\Delta_{n-1}(H)$$

where

$$D_n(H) = H_n'^{2}(x) - H_{n+1}'(x)H_{n-1}'(x)$$

For the Legendre polynomials,

Turán [56] observed that

$$\Delta_n(P) \geq 0 \quad \text{for} \quad |x| \leq 1$$

Danese [23] found explicitly

$$\Delta_n(P) = \frac{1-x^2}{n(n+1)} \sum_{i=o}^{n-1} \sum_{j=i}^{n-1} \frac{2i+1}{j+1} [P_i(x)]^2 \quad , \quad n \geq 1$$

Eweida [27] proved that

$$\text{(i)} \quad \Delta_n(P) = \frac{1-x^2}{n(n+1)} [1 + \sum_{\nu=2}^{n} \nu P_\nu^2(x) + (1-x^2) \sum_{\nu=2}^{n} \frac{\{P_\nu'(x)\}^2}{\nu}] \quad , \quad n > 1$$

(ii) $\Delta_n(P) < 0$ for all $x > 1$ or $x < -1$

(iii) $\lim_{x \to \pm 1} \left(\frac{\Delta_n(P)}{1-x^2}\right) = \frac{1}{2}$

Eweida [28] also proved that

$$\left\{\frac{d^r}{dx^r} P_n(x)\right\}^2 - \left\{\frac{d^r}{dx^r} P_{n-1}(x)\right\}\left\{\frac{d^r}{dx^r} P_{n+1}(x)\right\} > 0$$

$$r = 1,2,3,\ldots,(n-1) .$$

for all $n \geq 1$ and $-1 \leq x \leq 1$.

For the general Laguerre polynomials,

Danese [23] proved that

$$\Delta_n(L^{(\alpha)}) = \frac{\Gamma(n+\alpha)}{(n+1)!} \sum_{k=o}^{n} \frac{k!}{\Gamma(k+\alpha)} [L_k^{(\alpha-1)}(x)]^2 , \quad n \geq 1 , \quad \alpha > 0$$

Al-Salam [2] proved that

$$\Delta_n\left(\frac{dk}{dx^k} L^{(\alpha)}\right) \geq 0 \quad \text{for} \quad \alpha > -k - 1 \quad \text{and} \quad -\infty < x < \infty$$

Eweida [27] proved that

$$\text{(i)} \quad \Delta_n(L) = \frac{x^2}{n(n+1)} \sum_{\nu=1}^{n} \frac{\{L'_\nu(x)\}^2}{\nu} \quad \text{for all} \quad x \neq 0$$

$$\text{(ii)} \quad \lim_{x \to o} \frac{\Delta_n(L)}{x^2} = \frac{1}{2}$$

$$\text{(iii)} \quad \Delta_n(L) \to \infty \quad \text{as} \quad x \to \infty \quad \text{or} \quad x \to -\infty$$

where $L_n(x)$ is the simple Laguerre polynomial of degree n

For the Bessel functions, the Turán inequality follows immediately from the following identity:

$$[J_\mu(t)]^2 - J_{\mu+1}(t)J_{\mu-1}(t) = \frac{4}{t^2} \sum_{n=o}^{\infty} (2n+\mu+1)[J_{2n+\mu+1}(t)]^2$$

where $J_\mu(t)$ is the Bessel function of order μ, when $\mu > 0$ and t real. The above result was derived by Lommel [58, p. 152].

Al-Salam [2] proved that for the polynomials $\{p_n(x)\}$, which are defined by

$$p_{n+1}(x) = (A_n x+B_n)p_n(x) - C_n p_{n-1}(x) \quad (n \geq 1) \quad B_n = 0$$

where $\quad C_n > 0 \, , \quad A_n > 0 \, , \quad C_n A_{n-1} - A_n C_{n-1} \geq 0 \, , \quad A_n \leq A_{n-1}$

and $$\lambda_{n,k}(x) = p_n^{(k)}(x)p_{n-1}^{(k-1)}(x) - p_n^{(k-1)}(x)p_{n-1}^{(k)}(x) > 0$$

$\Delta_n(p^{(k)})$, where $p^{(k)}$ denotes the kth derivative, satisfies the following relation

$$\frac{A_{n-1}}{A_n}\Delta_n(p^{(k)}) = C_{n-1}\Delta_{n-1}(p^{(k)}) + kA_{n-1}\lambda_{n,k}(x)$$
$$+ \left(\frac{A_{n-1}C_n - A_nC_{n-1}}{A_n}\right)(p_{n-1}^{(k)}(x))^2 + \left(\frac{A_{n-1}}{A_n} - 1\right)(p_n^{(k)}(x))^2 .$$

Carlitz [14] proved the Turán inequality for the inverted Bessel polynomials $\{\theta_n(x)\}$, defined as

$$\theta_n(x) = x^n y_n(1/x) ,$$

where

$$y_n(x) = \sum_{r=o}^{n} \frac{(n+r)!}{(n-r)!}\left(\frac{x}{2}\right)^r$$

His result is that

$$\theta_n(x)\theta_{n+2}(x) - \theta_{n+1}^2(x) = x^{2n+1} + 2\sum_{j=o}^{[n/2]}(2n+1-4j)x^{4j}\theta_{n-2j}^2(x) .$$

Different types of generalizations of the Turán expression

have been considered by several authors and we shall state in brief the results proved by them.

Thiruvenkatachar and Nanjudiah [53] considered the expression

$$\Delta_n^{(k)} f = f_{n+k-1}(x) f_{n-k+1}(x) - f_{n+k}(x) f_{n-k}(x)$$

which reduces to $\Delta_n(f)$ when $k = 1$. They proved that

(i) $$0 \leq \Delta_n^{(k)}(I) \leq \frac{2k-1}{n+k} I_{n-k+1}(x) I_{n+k-1}(x)$$

where $1 \leq k \leq (n+2)$ and $I_n(x)$ is the modified Bessel function of the first kind.

(ii) $$\Delta_n^{(k)}(P) \leq 0 \ , \quad 1 \leq k \leq n \ , \quad |x| > 1$$

where $P_n(x)$ is the nth Legendre polynomial.

(iii) $$\Delta_n^{(k)}(L) \geq 0 \ , \quad 1 \leq k \leq n \ , \quad x < 0$$

where $L_n(x)$ is the simple Laguerre polynomial of the nth degree.

Now let us define

$$\Delta_{n,k}(f) = f_n^2(x) - f_{n+k}(x) f_{n-k}(x)$$

so that

$$\Delta_{n,k}(f) = \sum_{j=1}^{j=k} \Delta_n^{(j)}(f) .$$

If is easy to see that the following are true due to the above results.

$$0 \leq \Delta_{n,k}(I) \leq \sum_{i=1}^{i=k} \frac{2i-1}{n+i} I_{n+i-1}(x) I_{n-i+1}(x) ,$$

$$1 \leq k \leq (n+2)$$

$$0 \leq \Delta_{n,k}(L) , \quad 1 \leq k \leq n , \quad x < 0$$

and

$$0 > \Delta_{n,k}(P) , \quad n \geq 1 , \quad k \geq 1 , \quad (n-k) \geq 0 \text{ and } |x| > 1 ,$$

where I_k , L_k , and P_k are the modified Bessel function of the first kind, the simple Laguerre and the Legendre polynomials respectively.

Beckenbach, Seidel and Szász [11] proved that

$$\Delta_{n,k}(P) < 0 \text{ for } n \geq 1 , \quad k \geq 1 , \quad n - k \geq 0 \text{ and } |x| > 1$$

with the help of convexity properties of the function $|P_n(x)|$ for

$|x| > 1$, where $P_n(x)$ is the Legendre polynomial of the nth degree.

Let us define the following function of the real variable x :

$$\Delta_{n,h,k;x}(f) = f_{n+h}(x)f_{n+k}(x) - f_n(x)f_{n+h+k}(x)$$

for a function $f_n(x)$, where h and k are integers such that $k \geq h \geq 1$ so that $\Delta_{n,1,1;x}(f) = \Delta_{n+1}(f)$.

Forsythe proved [30] for Legendre polynomial the following inequalities:

$$\Delta_{n,1,2;x}(P) > 0 \quad \text{for all } x \text{ in } (0,1) \quad \text{for } n \geq 0$$

$$\Delta_{2n-1,2,2;x}(P) > 0 \quad \text{for all } x \text{ in } (0,1) \quad \text{for } n \geq 1 .$$

Chatterjea [15] proved

$$\left.\begin{array}{lr} \Delta_{n,1,2;x}(J) < 0 & -\infty < x < 0 \\ \qquad\quad = 0 & x = 0 \\ \qquad\quad > 0 & 0 < x < \infty \end{array}\right\} \quad \text{for } n > 0$$

where J_n is the Bessel function of the 1st kind and of order n , for $n > -1$ and $-\infty < x < \infty$. The method used is of recurrence relation. It is also proved that

$$3 \sum_{i=o}^{n-1} \Delta_{i,1,2;x}(J) \gtrless n \cdot \Delta_{n,1,2;x}(J)$$

according as $0 < x < \infty$ or $-\infty < x < 0$.

For the Hermite polynomials, Chatterjea [19] , proved

$$\Delta_{n,1,2;x}(H) \begin{matrix} < 0 \\ = 0 \\ > 0 \end{matrix} \quad \left.\begin{matrix} x < 0 \\ x = 0 \\ x > 0 \end{matrix}\right\} \qquad \text{for } n \geq 0$$

and

$$\Delta_{n,1,2,x}\left(\frac{d^k}{dx^k}(H)\right) \begin{matrix} < 0 \\ = 0 \\ > 0 \end{matrix} \quad \left.\begin{matrix} x < 0 \\ x = 0 \\ x > 0 \end{matrix}\right\} \qquad 1 \leq k \leq n .$$

Khandekar [36] found the explicit evaluations of $\Delta_{n,1,k;x}$ for Ultraspherical polynomials and Ultraspherical polynomials in the normalised form.

Danese [24] gave some explicit evaluations of certain Turán expressions. He proved:

(i) If $p_n^{(\lambda)}(x) = \dfrac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)}$, where $P_n^{(\lambda)}(x)$ is the Ultraspherical polynomial, then

$$\Delta_{n,1,2;x}(p^{(\lambda)}) = \frac{16\lambda n!(n+2)!x(1-x^2)}{\Gamma(n+2\lambda+1)\Gamma n+2\lambda+3)} \sum_{i=o}^{n} \sum_{j=o}^{[\frac{1}{2}(n-i)]} \frac{(\lambda+i)\Gamma(2\lambda+i)(\lambda+i+2j+1)}{i!(i+2j+2)!} \times$$

$$\times \Gamma(2\lambda+i+2j)[p_i^{(\lambda)}(x)]^2$$

$$\lambda \neq \frac{-m}{2}\ , \quad m = 0,1,2,3,4,\ldots$$

$$n \geq 0 \qquad -1 \leq x \leq 1$$

(ii) for the Legendre polynomial $P_n(x)$, $(\lambda = \frac{1}{2})$

$$\Delta_{n,1,2;x}(P) = P_{n+1}(x)P_{n+2}(x) - P_n(x)P_{n+3}(x)$$

$$= \frac{2x(1-x^2)}{(n+1)(n+3)} \sum_{i=o}^{n} \sum_{j=o}^{[\frac{1}{2}n-1]} \frac{(2i+1)(2i+4j+3)}{(i+2j+2)(i+2j+1)} [P_i(x)]^2$$

(iii) for the Tchebychef polynomial of the first kind $T_n(x)$, $(\lambda = 0)$

$$\Delta_{n,1,2;x}(T) = T_{n+1}(x)T_{n+2}(x) - T_n(x)T_{n+3}(x) = 2x(1-x^2)$$

(iv) for the Hermite polynomial

$$H_n(x) = \lim_{\lambda\to\infty} \lambda^{\frac{1}{2}n} P_n^{\frac{1}{2}\lambda} (x/\sqrt{\lambda})$$

$$\Delta_{n,1,2;x}(H) = H_{n+1}(x)H_{n+2}(x) - H_{n+3}(x)H_n(x) = 2xn! \sum_{i=o}^{[n/2]} \frac{H^2_{n-2i}(x)}{(n-2i)!}$$

This last result was also obtained by Toscano [41] .

Another type of generalization is of the form

$$\Omega_n^{(m)}(f) = \sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} f_{n+r}(x) f_{n-r}(x)$$

which reduces to $\Delta_n(f)$ when $m = 1$.

Toscano [42] proved the following result for the Hermite polynomials

$$\Omega_n^{(m)}(H) = \sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} H_{n+r}(x) H_{n-r}(x) = \frac{(2m)!(n-m)!}{m!} \sum_{k=m}^{n} \binom{k-1}{m-1} \frac{H_{n-k}^2(x)}{(n-k)!}$$

$$(1 \leq m \leq n)$$

Later on, Al-Salam and Carlitz [8] obtained similar kinds of results for the Hermite, Laguerre and Ultraspherical polynomials. In [4] , Al-Salam proved the result for the Bessel functions of the 1st kind:

$$\Omega_n^{(m)}(J) = \sum_{r=-m}^{m} (-1)^r \binom{2m}{m-r} J_{n-r}(x) J_{n+r}(x)$$

$$= \frac{4^m (2m)!}{x^{2m} m! (m-1)!} \sum_{k=0}^{\infty} (n+m+2k)(k+1)_{m-1} (n+k-1)_{m-1} J_{n+m+2k}^2(x)$$

where $(a)_m = a(a+1)(a+2)\ldots(a+m-1)$, $(a)_0 = 1$, and J_n is the Bessel function of order n .

In [6], Al-Salam found out the generating function for the generalized Turán expression for the Hermite Polynomials

$$\sum_{n=o}^{\infty} \Omega_{n+m}^{(m)} (H) \frac{t^n}{n!} = \frac{(2m)!}{m!(1-t)^m \sqrt{(1-t^2)}} \cdot e^{\frac{x^2 t}{1+t}}$$

Al-Salam has also shown the generating function for $\Delta_{n,1,2;x}(H)$

$$\sum_{n=o}^{\infty} \frac{\Delta_{n,1,2;x}(H)}{n!} t^n = \frac{2x}{(1-t^2)^{3/2}} \cdot e^{\frac{x^2 t}{1+t}}$$

Later on, Chatterjea [18], proved a generalized Turán expression for the Hermite polynomials

$$\Delta_{n+r,1,2;x}^{(r)}(H) = \frac{(n+r)!(n+r+2)!}{(n+3)!} 2x\{(n+3) \sum_{i=o}^{[n/2]} \frac{H^2_{n-2i}(x)}{(n-2i)!} + r \sum_{i=o}^{[n+\frac{1}{2}]} \frac{H^2_{n+1-2i}(x)}{(n+1-2i)}\}$$

and the generating function is given by

$$\sum_{n=o}^{\infty} \Delta_{n+r,1,2;x}^{(r)}(H) \frac{(n+3)!}{(n+r)!(n+r+2)!} t^{n+r+2} = 2x\{\frac{d}{dt} (\frac{t^{r+3}}{(1-t^2)} \cdot \frac{e^{\frac{x^2 t}{1+t}}}{(1-t^2)^{\frac{1}{2}}})$$

$$+ r \frac{t^{r+1}}{(1-t^2)} (\frac{1}{(1-t^2)^{\frac{1}{2}}} e^{x^2 t/1+t} - 1)\} ;$$

$$|t| < 1$$

where

$$\Delta^{(r)}_{n,1,2;x}(H) = \frac{d^r}{dx^r}\{H_{n+1}(x)\}\frac{d^r}{dx^r}\{H_{n+2}(x)\} - \frac{d^r}{dx^r}\{H_n(x)\}\frac{d^r}{dx^r}\{H_{n+3}(x)\} \quad .$$

One of the polynomials as discussed by Carlitz [10] is

$$H_n(x,q) = \sum_{r=o}^{n} [{n \atop r}]x^r$$

where

$$[{n \atop r}] = \frac{(1-q^n)(1-q^{n-1})(1-q^{n-2})\dots(1-q^{n-r+1})}{(1-q)(1-q^2)\dots(1-q^r)} \; , \; [{n \atop o}] = 1 \; .$$

Al-Salam and Carlitz [7] have proved a result, a q-analogue of a formula of Toscano

$$\text{(A)} \qquad \begin{aligned} \Omega^{(m)}_n(x,q) &= \sum_{r=-m}^{m} (-1)^r [{2m \atop m-r}] q^{\frac{1}{2}r(r+1)} H_{n+r}(x,q) H_{n-r}(x,q) \\ &= \frac{(q)_{2m}(q)_{n-m}}{(q)_m} \sum_{k=m}^{n} [{k-1 \atop m-1}] q^{(n-k)m} x^k \frac{H^2_{n-k}(x,q)}{(q)_{n-k}} \end{aligned}$$

where

$$(q)_m = (1-q)(1-q^2)(1-q^3)\dots(1-q^m) \; , \quad (q)_o = 1$$

$$(a)_m = (1-a)(1-aq)(1-aq^2)\dots(1-aq^{m-1}) \; , \quad (a)_o = 1 \quad \text{and} \quad (1 \le m \le n)$$

Again, defining,

$$G_n(x,q) = H_n(x,q^{-1})$$

the above result becomes

$$\sum_{r=-m}^{m} (-1)^r \begin{bmatrix} 2m \\ m-r \end{bmatrix} q^{\frac{1}{2}r(r-1)} G_{n+r}(x,q) G_{n-r}(x,q)$$

(B)

$$= q^{-m} \frac{(q)_{2m}(q)_{n-m}}{(q)_m} \sum_{k=m}^{n} (-1)^k \begin{bmatrix} k-1 \\ m-1 \end{bmatrix} \frac{q^{\frac{1}{2}k(k+1-2n)} x^k G_{n-k}^2(x,q)}{(q)_{n-k}} .$$

In particular when $m = 1$, (A) and (B) become

$$H_n^2(x,q) - H_{n+1}(x,q)H_{n-1}(x,q) = (1-q)(q)_{n-1} \sum_{k=1}^{n} \frac{q^{n-k} x^k H_{n-k}^2(x,q)}{(q)_{n-k}}$$

and

$$G_n^2(x,q) - G_{n+1}(x,q)G_{n-1}(x,q) = q^{-1}(1-q)(q)_{n-1} \sum_{k=1}^{n} (-1)^k q^{\frac{1}{2}k(k+1-2n)} \frac{G_{n-k}^2(x,q)}{(q)_{n-k}}$$

Thus the Turán expressions are proved for the polynomials defined above and these can be proved also with recurrence relations.

If $m = n$, we get the following interesting result

$$\sum_{r=-n}^{n} (-1)^r \left[{2n \atop n-r}\right] q^{\frac{1}{2}r(r+1)} H_{n+r}(x,q) H_{n-r}(x,q) = \frac{(q)_{2m}}{(q)_n} x^n$$

$$\sum_{r=-n}^{n} (-1)^r \left[{2n \atop n-r}\right] q^{\frac{1}{2}r(r-1)} G_{n+r}(x,q) G_{n-r}(x,q) = (-1)^n q^{-\frac{1}{2}n(n+1)} \frac{(q)_{2n}}{(q)_n} x^n \quad .$$

Again, Al-Salam [6] has obtained the generating function of a q-analogue of $\Omega_n(x)$ as

$$\sum_{n=o}^{\infty} \frac{\Omega^{(m)}_{n+m}(x,q)\, t^n}{[n]!} = \frac{[2m]!\, x^m}{[m]!\,(tx)_m} \prod_{r=o}^{\infty} \frac{1-q^{r+2m} x^2 t^2}{(1-q^{r+m}t)(1-q^{r+m}xt)^2 (1-q^{r+m}x^2 t)}$$

CHAPTER II

SEVERAL METHODS FOR PROVING THE TURÁN INEQUALITY

In this chapter we are going to discuss various methods adopted by different authors in proving the Turán inequality for different classical orthogonal polynomials and the Bessel functions of the ordinary and modified form.

As already pointed out the Turán inequality was observed by P. Turán while discussing the properties of the zeros of the Legendre polynomials and therefore it is essential to discuss Turán's method in detail in proving the inequality for the Legendre polynomials.

1. P. Turán [56] proved the following result for the Legendre polynomial.

If $x_{v,n}$ $(v = 1,2,...,n)$ denote the zeros of $P_n(x)$ and $x_{v,n-1}$ $(v = 1,2,...,(n-1))$ the zeros of $P_{n-1}(x)$, then

$$x_{1,n} - x_{1,n-1} < x_{2,n} - x_{2,n-1} < \dots < x_{[\frac{1}{2}(n-1)],n} - x_{[\frac{1}{2}(n-1)],n-1}$$

In proving the above theorem, Turán used the following lemma:

Lemma: $\Delta_n(P)$ is monotonically decreasing in the interval

$$1/(2n+1) \leq x \leq 1$$

Proof: We know the following recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

then

(2.1) $$(n+1)\Delta_n(P) = (n+1)P_n^2(x) - (2n+1)xP_n(x)P_{n-1}(x) + nP_{n-1}^2(x) .$$

Differentiating with respect to x, we have

$$(n+1)\Delta_n'(P) = 2(n+1)P_n(x)P_n'(x) - (2n+1)P_n(x)P_{n-1}(x) - (2n+1)xP_n'(x)P_{n-1}(x) -$$

$$- (2n+1)xP_n(x)P_{n-1}'(x) + 2nP_{n-1}(x)P_{n-1}'(x)$$

Since

$$P_n(x) = \frac{x}{n} P_n'(x) - \frac{1}{n} P_{n-1}'(x)$$

and

$$P_{n-1}(x) = \frac{1}{n} P_n'(x) - \frac{x}{n} P_{n-1}'(x) .$$

For the above relations see [52, p. 84] . By employing the above relations, we have

$$(n+1)\Delta_n'(P) = -\frac{(n+1)}{n^2} P_n'^2(x) - \frac{(n+1)}{n^2} P_{n-1}'^2(x) + \frac{1+(2n+1)x^2}{n^2} P_n'(x)P_{n-1}'(x)$$

which is a negative definite quadratic form of $P_n'(x)$ and $P_{n-1}'(x)$ if the discriminant is ≤ 0. But this is apart from a positive factor

$$[1+(2n+1)x^2]^2 - 4(n+1)^2x^2 = (1-x)[1-(2n+1)x][1+2(n+1)x+(2n+1)x^2]$$

which is ≤ 0 for

$$1/(2n+1) \leq x \leq 1$$

<u>Theorem 1.</u> $\Delta_n(P) \geq 0$ <u>for</u> $n = 1,2,\ldots$ <u>and</u> $-1 \leq x \leq 1$.

Proof: Since $\Delta_n(P)$ is an even function, it is sufficient to consider $0 \leq x \leq 1$. Using the recurrence relation, we have

$$(n+1)\Delta_n(P) = (n+1)P_n^2(x) - (2n+1)xP_n(x)P_{n-1}(x) + nP_{n-1}^2(x) \quad .$$

It is a positive definite form if the discriminant is ≤ 0 i.e.

$$(2n+1)^2x^2 - 4n(n+1) \leq 0$$

or

$$0 \leq x \leq 2\sqrt{n(n+1)}/2n+1 \quad . \tag{2.2}$$

But $\Delta_n(1) = 0$, thus the lemma gives immediately the positivity for

(2.3) $$1/2n+1 \leq x \leq 1$$

Obviously the intervals (2.2) and (2.3) cover for $n \geq 1$ the whole interval $0 \leq x \leq 1$. Hence the result is proved.

In the case of the Hermite polynomials, a similar method was used by Toscano [54] . He first shows that $\Delta'_n(H)$ is a monotonically increasing function for real $x \geq 0$. For this, he proves that,

$$\Delta'_n(H) = (n-1)\delta_{n-2}(H) \quad ,$$

where

$$\delta_n(H) = H_{n+1}(x)H_{n+2}(x) - H_n(x)H_{n+3}(x)$$

and

$$\delta_n(H) = 2xn! \sum_{i=o}^{[n/2]} \frac{H^2_{n-2i}(x)}{(n-2i)!}$$

Thus $\Delta'_n(H) \geq 0$ if $x \geq 0$. Hence $\Delta_n(H)$ is a monotonically increasing function for real $x \geq 0$.

Theorem 2. $\Delta_n(H)$ is ≥ 0 for all real x .

Proof: Since $H_n(-x) = (-1)^n H_n(x)$, therefore, $\Delta_n(H)$ is an even function of x , it is sufficient to consider only positive real x .

But

$$\Delta_n(H) = 2^{2n}[(\tfrac{1}{2})_{n/2}]^2 \quad \text{for} \quad x = 0 \quad \text{and} \quad n \quad \text{even}$$

$$= 2^{2n}(\tfrac{1}{2})_{\frac{n+1}{2}}(\tfrac{1}{2})_{\frac{n-1}{2}} \quad \text{for} \quad x = 0 \quad \text{and} \quad n \quad \text{odd.}$$

Thus $\Delta_n(H)$ is positive for all $n \geq 1$ and at $x = 0$, since it is an increasing function of real $x \geq 0$, thus the Turán inequality follows at once.

2. Szegö [50] has given four different methods for proving the Turán inequality for the Legendre polynomials and in his third proof, he has proved the Turán inequalities for the other classical orthogonal polynomials. Now we shall take each method in brief and separately.

(i) Method I. Szegö has proved the positiveness of the expression $\Delta_n(P)$ for $|x| \leq 1$ with the help of a recurrence relation and use of Mehler's formula. As far as the recurrence part is concerned it is exactly as Turán method and the range is $|x| < \frac{2(n(n+1))^{\frac{1}{2}}}{2n+1} = \cos\theta_o$. For these x the theorem is proved and for the remaining, put $x = \cos\theta$, that is for $0 < \theta < \theta_o$, we use Mehler's formula

$$P_n(\cos\theta) = \frac{2}{\pi}\int_o^{\theta} \frac{\cos(n+\frac{1}{2})u\,du}{(2(\cos u - \cos\theta))^{\frac{1}{2}}}$$

and obtain

$$\Delta_n(\cos\theta) = \pi^{-2}\int_o^\theta\int_o^\theta (\cos u - \cos\theta)^{-\frac{1}{2}}(\cos v - \cos\theta)^{-\frac{1}{2}}$$

$$\{2\cos(n+\tfrac{1}{2})u\cos(n+\tfrac{1}{2})v - \cos(n-\tfrac{1}{2})u\cos(n+\tfrac{3}{2})v - \cos(n-\tfrac{1}{2})v\cos(n+\tfrac{3}{2})u\}\,du\,dv$$

It is not difficult to see that the expression in the braces becomes

$$\cos(n+\tfrac{1}{2})(u+v)(1-\cos(u-v)) + \cos(n+\tfrac{1}{2})(u-v)(1-\cos(u+v))$$

so that $\Delta_n > 0$ follows provided

$$(n+\tfrac{1}{2})\,|u \pm v| \leq (n+\tfrac{1}{2})2\theta \leq (n+\tfrac{1}{2})\,2\theta_o \leq \frac{\pi}{2} \quad .$$

But this is obvious since

$$\theta_o < \frac{\pi}{2}\sin\theta_o = \frac{\pi}{2}\cdot\frac{1}{2n+1} \quad .$$

(ii) <u>Method II</u>. In this method $\Delta_n(P)$ has been expanded in a finite series of the Legendre polynomials which will contain only even terms i.e.

(2.4) $$\Delta_n(P) = c_oP_o(x) + c_1P_2(x) + c_2P_4(x) + \ldots + c_nP_{2n}(x)$$

and it has been shown that $c_1, c_2, c_3, \ldots, c_n$ are all negative with the help of a formula due to Adams, Ferrers and F. Neuman for the coefficients of the

Legendre expansion of the product of two Legendre polynomials. This formula is stated in the simplest form as

$$i(a,b,c) = \int_{-1}^{1} P_a P_b P_c \, dx = \begin{cases} 0 \text{ if } a+b+c \text{ odd}, \\ 0 \text{ if } a+b+c \text{ even but no} \\ \text{triangle with sides } a,b,c \text{ exists.} \\ \dfrac{g_s}{2s+1} \cdot \dfrac{g_{s-a} \cdot g_{s-b} \cdot g_{s-c}}{g_s} \end{cases}$$

if $a + b + c = 2s$, s integer and a triangle with sides a,b,c, exists, where

$$g_s = \frac{1 \cdot 3 \cdot 5 \ldots (2s-1)}{2 \cdot 4 \cdot 6 \ldots 2s} \; ; \quad g_o = 1$$

Multiplying both sides of the expansion (2.4) by $P_{2m}(x)$ and integrating with respect to x with limits from -1 to 1, we get

$$\frac{2}{4m+1} c_m = i(n,n,2v) - i(n-1,n+1,2m)$$

$$= \frac{2}{2(n+m)+1} \left(\frac{g_m g_m g_{n-m}}{g_{n+m}} - \frac{g_{m-1} g_{m+1} g_{n-m}}{g_{n+m}}\right), \quad m \geq 1$$

but g_m/g_{m-1} is increasing so that c_m is negative.

Now $\Delta_n(P)$ is minimum if $P_m(x)$ is maximum, that is, for

$x = 1$. But $\Delta_n(1) = 0$, obviously the inequality follows.

(iii) Method III. Let

$$G(x,z) = e^{-\alpha z^2+\beta z}\Pi(1+\beta_n z)e^{-\beta_n z}$$

where α , β and β_n are functions of x and $\alpha \geq 0$, β and β_n are real and $\sum \beta_n^2$ is convergent.

Such entire functions are said to be of Pólya and Schür types. These functions are either polynomials with real zeros or the limit of such polynomials.

These functions have been studied by Pólya and Schür and it has been proved [45, pp. 96-97] that if

$$\sum_{n=o}^{\infty} \frac{p_n(x)z^n}{n!} = G(x,z)$$

where $p_n(x)$ are real, then there exists a sequence

$$G_n(x,z) = p_{no}(x) + \frac{p_{n1}(x)}{1!} z + \frac{p_{n2}(x)}{2!} z^2 + \ldots + \frac{p_{nk_n}(x)z^{k_n}}{k_n!} ,$$

of polynomials with known real zeros which converge uniformly to $G(x,z)$ in $|z| \leq \rho$ and $\lim\limits_{n \to \infty} p_{nv} = p_v$ $(v = 0,1,2,\ldots)$.

Now we shall prove that the Jensen polynomial

$$p_o(x) + \binom{n}{1}p_1(x)z + \binom{n}{2}p_2(x)z^2 + \ldots + \binom{n}{n-1}p_{n-1}(x)z^{n-1} + p_n(x)z^n$$

has only real zeros.

First of all we state Schur's composition theorem:

Schurs' Composition theorem.

If all zeros of the polynomial

$$f(x) = a_o + a_1x + a_2x^2 + \ldots + a_mx^m \quad (a_m \neq 0)$$

are real and all zeros of the polynomial

$$g(x) = b_o + b_1x + b_2x^2 + \ldots + b_nx^n \quad (b_n \neq 0)$$

are real and of the same sign, then all zeros of the polynomial

$$P(x) = a_ob_o + a_1b_1x + 2!a_2b_2x^2 + \ldots + k!a_kb_kx^k$$

where $k = \min(m,n)$ _are real. If_ $m \leq n$ _here and_ $a_ob_o \neq 0$, _then all zeros of_ $P(x)$ _are distinct_. As an application of this theorem, consider the polynomial

$$(1+z)^m = \sum_{\mu=o}^{m} \binom{m}{\mu} z^\mu$$

Obviously the zeros of this polynomial are all real and of the same sign. Since

$$G_n(x,z) = p_{no}(x) + p_{n1}\frac{(x)}{1!}z + \frac{p_{n2}(x)}{2!}z^2 + \ldots$$

has only real zeros, it follows by the composition theorem that

$$\sum_{\mu=o}^{m} \binom{m}{\mu} p_{n\mu}(x) z^\mu = 0$$

has only real zeros.

Taking the limit as n tends to ∞ , we have

$$\sum_{\mu=o}^{m} \binom{m}{\mu} p_\mu(x) z^\mu = 0$$

has only real zeros, i.e.

$$\binom{m}{o} p_o(x) + \binom{m}{1} p_1(x) z + \binom{m}{2} p_2(x) z^2 + \ldots + p_m(x) z^m = 0$$

has only real zeros.

Theorem 3. *If $\{p_n(x)\}$ has the generating function of the following form*

$$\sum_{n=o}^{\infty} \frac{p_n(x)z^n}{n!} = G(x,z)$$

where $G(x,z)$ is an entire function as defined above, then the sequence $\{p_n(x)\}$ satisfies the Turán inequality.

Proof: It has been seen above that if $G(x,z)$ is an entire function of Pólya and Schür type, then the Jensen polynomial

$$\binom{n}{o}p_o(x) + \binom{n}{1}p_1(x)z + \ldots + \binom{n}{n-1}p_{n-1}(x)z^{n-1} + p_n(x)z^n$$

has only real zeros.

If we denote the zeros of this polynomial by $z_1, z_2, \ldots, z_n$, then

$$\sum_i z_i = z_1 + z_2 + \ldots + z_n = -\frac{\binom{n}{1}p_{n-1}(x)}{p_n(x)}$$

$$\sum_{i \neq j} z_i z_j = z_1z_2 + z_2z_3 + \ldots = \frac{\binom{n}{2}p_{n-2}(x)}{p_n(x)} \quad , \quad 1 \leq i \; , \; j \leq n \; .$$

If the zeros are real, then the following inequality is always true:

$$(2.5)\qquad \left\{\frac{z_1+z_2+\ldots+z_n}{\binom{n}{1}}\right\}^2 \geq \frac{z_1z_2+z_2z_3+\ldots}{\binom{n}{2}}$$

Obviously the Turán inequality $\Delta_{n-1}(P) \geq 0$ follows from the above inequality, and the above expressions for $\sum_i z_i$ and $\sum_{i\neq j} z_i z_j$.

Examples. (i) For the Legendre polynomials, the generating function is given by

$$\sum_{n=o}^{\infty} \frac{P_n(x)z^n}{n!} = e^{xz} J_o((1-x^2)^{\frac{1}{2}}z)$$

In the case of the Bessel function J_o , all zeros are real and may be expressed as [58, p. 498]

$$J_o(z) = \prod_{n=1}^{\infty} \{(1-\frac{z}{j_{o,n}})e^{z/j_{o,n}}\} \prod_{n=1}^{\infty} (1+\frac{z}{j_{o,n}})e^{-z/j_{o,n}}\}$$

where $\pm j_{o,n}$ $(n = 1,2,3,\ldots)$ are the zeros of $J_o(z)$.

Thus the conditions of the theorem are satisfied and hence the Turán inequality for the Legendre polynomials follows.

(ii) The generating functions for the Ultraspherical, the general Laguerre and the Hermite polynomials are given respectively by

[illegible]

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$$\sum_{n=o}^{\infty} \frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)} \frac{z^n}{n!} = 2^{\lambda-\frac{1}{2}}\Gamma(\lambda+\frac{1}{2})e^{xz}((1-x^2)^{\frac{1}{2}}z)^{\frac{1}{2}-\lambda} J_{\lambda-\frac{1}{2}}((1-x^2)^{\frac{1}{2}}z), \quad \lambda > -\frac{1}{2}$$

$$\sum_{n=o}^{\infty} \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \frac{z^n}{n!} = \Gamma(\alpha+1)(xz)^{-\alpha/2} J_\alpha(2(xz)^{\frac{1}{2}}), \quad \alpha > -1$$

$$\sum_{n=o}^{\infty} \frac{H_n(x)}{n!} z^n = e^{2xz-z^2}$$

Since we know that $J_v(z)$, $v > -1$, has no zeros which are not real and if $\pm j_{v,1}, \pm j_{v,2}, \ldots$ are the real zeros of $z^{-v}J_v(z)$, then it can be expressed as [58, p. 498]

$$z^{-v}J_v(z) = \frac{(\frac{1}{2})^v}{\Gamma(v+1)} \prod_{n=1}^{\infty} \{(1-\frac{z}{j_{v,n}})e^{z/j_{v,n}}\} \prod_{n=1}^{\infty} \{(1+\frac{z}{j_{v,n}})e^{-z/j_{v,n}}\}$$

or

$$z^{-v}J_v(z) = \frac{(\frac{1}{2})^v}{\Gamma(v+1)} \prod_{n=1}^{\infty} (1-\frac{z^2}{j_{v,n}^2}) \quad \text{where} \quad v > -1$$

Thus the generating functions for the above polynomials satisfy the conditions as mentioned in the theorem 3 .

Hence the Turán inequalities follow in the above cases.

(iv) <u>Method IV</u>. This method does not deal with the generating function but deals with the Jensen polynomials only.

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For the Legendre polynomials, we have the following identity

$$P_o(x) + \binom{n}{1}P_1(x)z + \binom{n}{2}P_2(x)z^2 + \ldots + P_n(x)z^n = (1+2xz+z^2)^{n/2}P_n\left(\frac{1+xz}{(1+2xz+z^2)^{\frac{1}{2}}}\right)$$

It can be easily established either by the ordinary generating function or by the first integral of Laplace for the Legendre polynomial. We shall prove the above identity with the help of the first integral of Laplace.

The first integral of Laplace is given by

$$P_n(x) = \pi^{-1}\int_o^{\pi} \{x+(x^2-1)^{\frac{1}{2}}\cos\phi\}^n d\phi$$

Now

$$\begin{aligned}
\sum_{k=o}^{n} \binom{n}{k}P_k(x)z^k &= \pi^{-1}\sum_{k=o}^{n}\binom{n}{k}z^k\int_o^{\pi}\{x+(x^2-1)^{\frac{1}{2}}\cos\phi\}^k d\phi \\
&= \pi^{-1}\int_o^{\pi}\sum_{k=o}^{n}\binom{n}{k}z^k\{x+(x^2-1)^{\frac{1}{2}}\cos\phi\}^k d\phi \\
&= \pi^{-1}\int_o^{\pi}\{1+xz+(x^2-1)^{\frac{1}{2}}\cos\phi z\}^n d\phi \\
&= (1+2xz+z^2)^{\frac{n}{2}}\pi^{-1}\int_o^{\pi}\left\{\frac{1+xz}{(1+2xz+z^2)^{\frac{1}{2}}}+\frac{(x^2-1)^{\frac{1}{2}}\cos\phi z}{(1+2xz+z^2)^{\frac{1}{2}}}\right\}^n d\phi \\
&= (1+2xz+z^2)^{\frac{n}{2}}P_n\left(\frac{1+xz}{(1+2xz+z^2)^{\frac{1}{2}}}\right)
\end{aligned}$$

Let x be fixed, $-1 < x < 1$. We obtain for the roots of the polynomial in z the condition

$$\frac{1+xz}{(1+2xz+z^2)^{\frac{1}{2}}} = x_\nu$$

where x_ν denotes a root of P_n, or

$$z = \frac{x(x_\nu^2-1) \pm x_\nu((1-x_\nu^2)(1-x^2))^{\frac{1}{2}}}{x^2 - x_\nu^2}$$

thus the roots are all real. Using the inequality (2.5), the result follows.

In [52] Szegö has not indicated that the Turán inequality for the Hermite, the general Laguerre and the Ultraspherical polynomials can be proved with this method. We shall furnish the proof of each of these cases separately.

(i) <u>Hermite polynomials</u>

The Hermite polynomial is defined as

$$H_n(x) = (2x)^n {}_2F_0(-n/2,-n/2+\tfrac{1}{2};--;-1/x^2) \tag{2.6}$$

Consider

$$\sum_{n=0}^{n=N} \frac{H_n(x)}{n!}(-N)_n t^n = \sum_{n=0}^{n=N} \frac{t^n(-N)_n}{n!} \sum_{k=0}^{[n/2]} \frac{(-1)^k n!(2x)^{n-2k}}{k!(n-2k)!}$$

$$0 \leq k \leq \frac{n}{2} \leq \frac{N}{2}$$

$$= \sum_{k=o}^{[N/2]} \frac{(-1)^k}{k!} \sum_{n=2k}^{N} \frac{t^n(-N)_n}{(n-2k)!} (2x)^{n-2k}$$

$$= \sum_{k=o}^{[N/2]} \frac{(-1)^k}{k!} \sum_{n=o}^{N-2k} \frac{t^{n+2k}(-N)_{n+2k}(2x)^n}{n!}$$

$$= \sum_{k=o}^{[N/2]} \frac{(-1)^k(-N)_{2k}}{k!} t^{2k} \sum_{n=o}^{N-2k} \frac{(2xt)^n(-N+2k)_n}{n!}$$

$$= \sum_{k=o}^{[N/2]} \frac{(-1)^k(-N)_{2k}t^{2k}}{k!(1-2xt)^{-N+2k}}$$

$$= (1-2xt)^N \sum_{k=o}^{[N/2]} (-1)^k \frac{2^{2k}(\frac{-N}{2})_k(\frac{-N}{2}+\frac{1}{2})_k t^{2k}}{k!(1-2xt)^{2k}}$$

$$= (1-2xt)^N {}_2F_o[\frac{-N}{2},\frac{-N}{2}+\frac{1}{2};--;\frac{-4t^2}{(1-2xt)^2}]$$

$$= (2t)^N(\frac{1}{2t}-x)^N {}_2F_o[\frac{-N}{2},\frac{-N}{2}+\frac{1}{2};--;\frac{-1}{(\frac{1}{2t}-x)^2}]$$

$$= (t)^N H_N[(\frac{1}{2t}-x)] \qquad \text{by } (2.6)$$

$$\therefore \quad \sum_{n=o}^{N} \binom{N}{n} H_n(x)t^n = (-t)^N H_N(-\frac{1}{2t}-x)$$

$$= t^N H_N(\frac{1}{2t}+x)$$

Let x be fixed. We obtain for the roots of the polynomial in t, the condition

$$\frac{1}{2t} + x = x_v$$

where x_v denotes a root of H_N. Or

$$\frac{1}{2t} = x_v - x$$

or

$$t = \frac{1}{2(x_v - x)}$$

Thus the roots in t are all real. Now using the inequality (2.5) we get $\Delta_{n-1}(H) \geq 0$.

(ii) <u>The general Laguerre polynomials</u>

$$(2.7) \qquad \sum_{k=o}^{n} \binom{n}{k} \frac{L_k^{(\alpha)}(x)}{L_k^{(\alpha)}(0)} z^k = \sum_{k=o}^{n} \frac{n!}{k!(n-k)!} z^k \sum_{j=o}^{k} \frac{(-k)_j x^j}{(1+\alpha)_j j!}$$

$$= n! \sum_{j=o}^{n} \frac{(-1)^j x^j}{j!(1+\alpha)_j} \sum_{k=j}^{n} \frac{z^k}{(n-k)!(k-j)!}$$

$$= n! \sum_{j=o}^{n} \frac{(-1)^j x^j z^j}{j!(1+\alpha)_j} \sum_{k=o}^{(n-j)} \frac{z^k}{(n-k-j)!k!}$$

$$= n! \sum_{j=o}^{n} \frac{(-1)^j x^j z^j}{j!(1+\alpha)_j (n-j)!} \sum_{k=o}^{(n-j)} \frac{(n-j)!}{(n-k-j)!} \frac{z^k}{k!}$$

$$= \sum_{j=o}^{n} \frac{(-n)_j (xz)^j}{j!(1+\alpha)_j} (1+z)^{n-j}$$

$$= (1+z)^n \sum_{j=o}^{n} \frac{(-n)_j}{j!(1+\alpha)_j} (\frac{xz}{1+z})^j$$

$$= (1+z)^n L_n^{\alpha}(\frac{xz}{1+z})/L_n^{(\alpha)}(0)$$

Let $f_r^{(\alpha)}(x) = \dfrac{L_r^{(\alpha)}(x)}{L_r^{(\alpha)}(0)}$, therefore (2.7) can be written as

$$\text{(2.8)} \qquad \sum_{k=o}^{n} \binom{n}{k} f_k^{(\alpha)}(x) z^k = (1+z)^n f_n^{(\alpha)}(\frac{xz}{1+z}) \quad \text{for} \quad \alpha > -1$$

Now let x be fixed, $0 \leq x < \infty$. We obtain for the roots of the polynomial (2.8) in z , the condition

$$\frac{xz}{1+z} = x_v ,$$

where x_v denotes a root of $f_n^{(\alpha)}(x)$ or

$$z = \frac{x_v}{x-x_v} .$$

Thus the roots in z are all real. If $z_1, z_2, \ldots, z_n$ are the roots of (2.8) , then by applying (2.5), we get

$$\left[\frac{-\binom{n}{n-1} f_{n-1}^{(\alpha)}(x)}{n f_n^{(\alpha)}(x)}\right]^2 \geq \frac{\binom{n}{n-2} f_{n-2}^{(\alpha)}(x)}{\binom{n}{2} f_n^{(\alpha)}(x)}$$

or

$$(f_{n-1}^{(\alpha)}(x))^2 \geq f_n^{(\alpha)}(x) f_{n-2}^{(\alpha)}(x) \quad \text{or} \quad \Delta_{n-1}\left(\frac{L^{(\alpha)}(x)}{L^{(\alpha)}(o)}\right) \geq 0$$

(iii) <u>Gegenbauer polynomials</u>

The Gegenbauer polynomial of the nth degree is defined as

$$C_n^{\nu}(x) = \frac{(2\nu)_n}{n\,!}\, x^n \; {}_2F_1\left(-\frac{1}{2}\,n, -\frac{1}{2}\,n + \frac{1}{2}; \nu + \frac{1}{2}; \frac{x^2-1}{x^2}\right)$$

Consider

$$(2.9) \quad \sum_{n=o}^{N} \frac{(-N)_n}{n!} Q_n^{\nu}(x) t^n = \sum_{n=o}^{N} \frac{(-N)_n t^n}{n!} \sum_{k=o}^{[n/2]} \frac{(-\frac{n}{2})_k (-\frac{n}{2}+\frac{1}{2})_k x^{n-2k} (x^2-1)^k}{k!\,(\nu+\frac{1}{2})_k}$$

where $Q_n^{\nu}(x) = \dfrac{n!C_n^{\nu}(x)}{(2\nu)_n}$. But $C_n^{\nu}(1) = \dfrac{(2\nu)_n}{n!}$, therefore, $Q_n^{\nu}(x) = \dfrac{C_n^{\nu}(x)}{C_n^{\nu}(1)}$

or

$$\sum_{n=o}^{N} \frac{(-N)_n}{n!} Q_n^{\nu}(x)t^n = \sum_{n=o}^{N} \frac{(-N)_n}{n!} t^n \sum_{k=o}^{[n/2]} \frac{(-n)_{2k}x^{n-2k}(x^2-1)^k}{2^{2k}k!(\nu+\frac{1}{2})_k}$$

$$0 \leq k \leq n/2 \leq N/2$$

$$= \sum_{k=o}^{[N/2]} \frac{(x^2-1)^k}{2^{2k}k!(\nu+\frac{1}{2})_k} \sum_{n=2k}^{N} \frac{(-N)_n t^n x^{n-2k}}{(n-2k)!}$$

$$= \sum_{k=o}^{[N/2]} \frac{(x^2-1)^k t^{2k}}{2^{2k}k!(\nu+\frac{1}{2})_k} \sum_{n=o}^{N-2k} (-N)_{n+2k} \frac{x^n t^n}{n!}$$

$$= \sum_{k=o}^{[N/2]} \frac{(x^2-1)^k t^{2k}(-N)_{2k}}{k!2^{2k}(\nu+\frac{1}{2})_k} \sum_{n=o}^{N-2k} (-N+2k)_n \frac{(xt)^n}{n!}$$

$$= \sum_{k=o}^{[N/2]} \frac{(x^2-1)^k t^{2k}(\frac{-N}{2})_k(\frac{-N}{2}+\frac{1}{2})_k}{k!(\nu+\frac{1}{2})_k} (1-xt)^{N-2k}$$

$$= (1-xt)^N \sum_{k=o}^{[N/2]} \frac{(\frac{-N}{2})_k(\frac{-N}{2}+\frac{1}{2})_k}{k!(\nu+\frac{1}{2})_k} \left[\frac{t^2(x^2-1)}{(1-xt)^2}\right]^k$$

$$= (1-xt)^N {}_2F_1\left[\frac{-N}{2},\frac{-N}{2}+\frac{1}{2};\nu+\frac{1}{2};\frac{t^2(x^2-1)}{(1-xt)^2}\right]$$

or

$$\sum_{n=o}^{N} \frac{N!}{(N-n)!n!} Q_n^{\nu}(x)(-t)^n = (1-xt)^N {}_2F_1\left[\frac{-N}{2},\frac{-N}{2}+\frac{1}{2};\nu+\frac{1}{2};\frac{t^2(x^2-1)}{(1-xt)^2}\right]$$

$$\sum_{n=o}^{N} \binom{N}{n} Q_n^{\nu}(x)t^n = (1+xt)^N {}_2F_1[\frac{-N}{2},\frac{-N}{2}+\frac{1}{2};\nu+\frac{1}{2};\frac{t^2(x^2-1)}{(1+xt)^2}]$$

$$= (1+xt)^N {}_2F_1[\frac{-N}{2},\frac{-N}{2}+\frac{1}{2};\nu+\frac{1}{2};1-\frac{1}{y^2}]$$

where $y = \dfrac{1+xt}{\sqrt{(1+2xt+t^2)}}$. But $Q_n^{\nu}(x) = x^n {}_2F_1[\frac{-n}{2},\frac{-n}{2}+\frac{1}{2};\nu+\frac{1}{2};1-\frac{1}{x^2}]$

$\therefore$

$$\sum_{n=o}^{N} \binom{N}{n} Q_n^{\nu}(x)t^n = [\sqrt{(1+2xt+t^2)}]^N Q_N^{\nu}\left(\frac{1+xt}{\sqrt{(1+2xt+t^2)}}\right) \tag{2.10}$$

Let x be fixed, $-1 < x < 1$. We obtain for the roots of the polynomial (2.10) in t the condition

$$\frac{1+xt}{\sqrt{1+2xt+t^2}} = x_\nu \quad ,$$

where x_ν denotes a root of $Q_N^{\nu}(x)$.

or
$$1 + x^2t^2 + 2xt = x_\nu^2(1+2xt+t^2)$$

or

$$t = \frac{-x(1-x_\nu^2) \pm x_\nu((1-x_\nu^2)(1-x^2))^{\frac{1}{2}}}{x^2-x_\nu^2}$$

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Thus the roots in t are all real. Using the trivial inequality, as before, the result follows.

<u>Corollary</u>. The Ultraspherical polynomial is given by

$$P_n^{(\alpha,\alpha)}(x) = \frac{(1+\alpha)_n}{(1+2\alpha)_n} C_n^{(\alpha+\frac{1}{2})}(x)$$

or

$$n! \frac{P_n^{(\alpha,\alpha)}(x)}{(1+\alpha)_n} = n! \frac{C_n^{(\alpha+\frac{1}{2})}(x)}{(2\alpha+1)_n}$$

or

$$\frac{P_n^{(\alpha,\alpha)}(x)}{P_n^{(\alpha,\alpha)}(1)} = \frac{C_n^{(\alpha+\frac{1}{2})}(x)}{C_n^{(\alpha+\frac{1}{2})}(1)} = Q_n^{(\alpha+\frac{1}{2})}(x) \quad .$$

Obviously the above result is the same for the Ultraspherical polynomial in the normalized form.

3. In 1946, Demir proposed the following result for the Hermite polynomials

$$H_{n+1}^2(x) - H_n(x)H_{n+2}(x) = n! \sum_{p=o}^{n} \frac{H_p^2(x)}{p!}$$

This result leads to the Turán inequality for the Hermite polynomials. This result was proved by the proposer [26] . The above problem was also solved with the help of the recurrence relation for the Hermite polynomials.

Eweida [27] , [28] has proved the Turán inequality for the Legendre, the Laguerre and the Ultraspherical polynomials with the help of recurrence formulae. His technique in applying the recurrence relation and proving the positiveness of the Turán expression is elementary.

Danese [23] has found the explicit evaluation of the Turán expression like Demir's result for the Ultraspherical, the Tchebichef, the general Laguerre and the Hermite polynomials. We shall state below the theorems as proved by the author.

Theorem 4. If $D_{n+1}^{(\lambda)}(x) = [\frac{d}{dx} P_{n+1}^{(\lambda)}(x)]^2 - \frac{d}{dx} P_n^{(\lambda)}(x) \frac{d}{dx} P_{n+2}^{(\lambda)}(x)$,

then

$$D_{n+1}^{(\lambda)}(x) = 4\lambda \sum_{i=o}^{n} \sum_{j=i}^{n} \frac{j+\lambda}{j+1} h_j^{(\lambda)} (h_i^{(\lambda)})^{-1} [P_i^{(\lambda)}(x)]^2 , \quad \lambda > -\frac{1}{2} , \quad \lambda \neq 0 , \quad n \geq 0 .$$

Theorem 5. If $\Delta_n^{(\lambda)}(x) = [P_n^{(\lambda)}(x)]^2 - P_{n+1}^{(\lambda)}(x) P_{n-1}^{(\lambda)}(x)$, then

$$\Delta_n^{(\lambda)}(x) = \frac{1}{\lambda-1} \sum_{i=o}^{n} \sum_{j=i}^{n} \frac{j+1}{j+\lambda-1} h_j^{(\lambda-1)} (h_i^{(\lambda-1)})^{-1} [P_i^{(\lambda-1)}(x)]^2 , \quad n \geq 1 , \lambda > \frac{1}{2} , \lambda \neq 1$$

where

$$h_k^{(\lambda)} = \frac{\sqrt{\pi}\ (2\lambda)_k \Gamma(\lambda+\frac{1}{2})}{(k+\lambda)k!\Gamma(\lambda)}$$

$$(2\lambda)_k = 2\lambda(2\lambda+1)\ldots(2\lambda+k-1)\ , \quad \lambda \neq 0$$

Theorem 6. If $F_n(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$, then

$$F_n^2(x) - F_{n+1}(x)F_{n-1}(x) = \frac{n!(n-1)!4\lambda(1-x^2)}{\Gamma(n+2\lambda)\Gamma(n+2\lambda+1)} \sum_{i=o}^{n-1} \sum_{j=i}^{n-1} \frac{(i+\lambda)(2\lambda)_j}{(j+1)!(2\lambda)_i} \frac{[\Gamma(i+2\lambda)]^2}{i!} [F_i(x)]^2$$

$$n \geq 1\ , \quad \lambda > -\frac{1}{2}\ , \quad \lambda \neq 0$$

The Tchebichef polynomial of the first kind is defined by

$$T_n(x) = \lim_{\lambda \to o} F_n(x)$$

Taking $\lambda \to 0$ in Theorem 5 yields the elementary identity

$$[T_n(x)]^2 - T_{n+1}(x)T_{n-1}(x) = 1 - x^2$$

or

$$\Delta_n(T) = \sin^2\theta \quad ,$$

where $x = \cos\theta$.

<u>Theorem 7.</u> $$\left[\frac{d}{dx} L_n^{(\alpha)}(x)\right]^2 - \frac{d}{dx} L_{n+1}^{(\alpha)}(x)\cdot\frac{d}{dx} L_{n-1}^{(\alpha)}(x)$$

$$= \frac{\Gamma(\alpha+n)}{n!} \sum_{k=o}^{n-1} \frac{k!}{\Gamma(\alpha+k+1)} [L_k^{(\alpha)}(x)]^2 , \quad n \geq 1 , \quad \alpha > -1 .$$

<u>Theorem 8.</u> $$[L_n^{(\alpha)}(x)]^2 - L_{n+1}^{(\alpha)}(x)L_{n-1}^{(\alpha)}(x) =$$

$$= \frac{\Gamma(n+\alpha)}{(n+1)!} \sum_{k=o}^{n} \frac{k!}{\Gamma(k+\alpha)} [L_k^{(\alpha-1)}(x)]^2 , \quad n \geq 1 , \quad \alpha > 0 .$$

<u>Theorem 9.</u> <u>If</u> $\Delta_n^{(\alpha)}(x) = [g_n^{(\alpha)}(x)]^2 - g_{n+1}^{(\alpha)}(x)g_{n-1}^{(\alpha)}(x)$, <u>where</u> $g_n^{(\alpha)}(x) = \dfrac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}$, <u>then</u>

$$\Delta_n^{(\alpha)}(x) = \frac{x^2(n-1)!}{\Gamma(\alpha+n+2)} \sum_{i=1}^{n} \frac{\Gamma(\alpha+i+1)}{i!i} \left[\frac{d}{dx} g_i^{(\alpha)}(x)\right]^2$$

<u>Theorem 10.</u> $$\left[\frac{d^r}{dx^r} H_n(x)\right]^2 - \frac{d^r}{dx^r} H_{n+1}(x) \cdot \frac{d^r}{dx^r} H_{n-1}(x)$$

$$= \frac{n!(n-1)!}{(n-r)!(n-r+1)!} \left[(n+1)(n-r)! \sum_{i=o}^{n-r-1} \frac{[H_i(x)]^2}{i!} + r[H_{n-r}(x)]^2\right]$$

$$n \geq 1 , \quad r \leq n - 1 .$$

4. Szász [48] and [49] has proved the Turán inequality for the general Laguerre, the Ultraspherical and the Bessel function. His

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technique for proving the results is unique. We shall treat each case separately.

(i) The Laguerre polynomials.

He considers the new function

$$g_n^{(\alpha)}(x) = L_n^{(\alpha)}(x)/L_n^{(\alpha)}(0) \cdot e^{-x/2}$$

and has established the following recurrence relation for the new function

$$(2.11) \qquad (n+\alpha+1)g_{n+1}^{(\alpha)}(x) = (2n+\alpha-1-x)g_n^{(\alpha)}(x) - ng_{n-1}^{(\alpha)}(x)$$

and

$$x\frac{d}{dx}g_n^{(\alpha)}(x) = (n-\frac{x}{2})g_n^{(\alpha)}(x) - ng_{n-1}^{(\alpha)}(x)$$

We put $\Delta_n^{(\alpha)}(x) = g_n^{(\alpha)^2}(x) - g_{n+1}^{(\alpha)}(x)g_{n-1}^{(\alpha)}(x)$. A simple calculation yields

$$x\frac{d}{dx}\Delta_n^{(\alpha)}(x) + \alpha\Delta_n^{(\alpha)}(x) = (g_{n+1}^{(\alpha)}(x)-g_n^{(\alpha)}(x))(g_n^{(\alpha)}(x)-g_{n-1}^{(\alpha)}(x))$$

or

$$\frac{d}{dx}(x^\alpha\Delta_n^{(\alpha)}(x)) = x^{\alpha-1}(g_{n+1}^{(\alpha)}(x)-g_n^{(\alpha)}(x))(g_n^{(\alpha)}(x)-g_{n-1}^{(\alpha)}(x))$$

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The function $f_\alpha(x) = x^\alpha \Delta_n^{(\alpha)}(x)$ vanishes at $x = 0$ and at $x = \infty$; at the points of the relative extrema either

$$(2.12) \qquad g_{n+1}^{(\alpha)}(x) = g_n^{(\alpha)}(x)$$

or

$$(2.13) \qquad g_n^{(\alpha)}(x) = g_{n-1}^{(\alpha)}(x)$$

Both cannot hold simultaneously as can be seen from the recurrence formula. If (2.12) holds, then

$$\Delta_n^{(\alpha)}(x) = g_n^{(\alpha)}(x)(g_n^{(\alpha)}(x)-g_{n-1}^{(\alpha)}(x))$$

Now from (2.11)

$$(n-x)g_n^{(\alpha)}(x) = ng_{n-1}^{(\alpha)}(x) \ , \quad n(g_n^{(\alpha)}(x)-g_{n-1}^{(\alpha)}(x)) = xg_n^{(\alpha)}(x)$$

Hence

$$\Delta_n^{(\alpha)}(x) = x/n \ (g_n^{(\alpha)}(x))^2 > 0$$

If (2.13) holds, then

$$\Delta_n^{(\alpha)}(x) = g_n^{(\alpha)}(x)(g_n^{(\alpha)}(x)-g_{n+1}^{(\alpha)}(x))$$

From (2.11)

$$(n+\alpha+1)g_{n+1}^{(\alpha)}(x) = (n+\alpha+1-x)g_n^{(\alpha)}(x) \ , \ (n+\alpha+1)(g_n^{(\alpha)}(x)-g_{n+1}^{(\alpha)}(x)) = xg_n^{(\alpha)}(x)$$

Hence

$$\Delta_n^{(\alpha)}(x) = x/n+\alpha+1 \ (g_n(x))^2 > 0$$

It follows that $\Delta_n^{(\alpha)}(x) > 0$ for $x > 0$.

(ii) The Ultaspherical polynomial.

In this case he considers the new function as $F_n(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$ and establishes the following recurrence relation

(2.14) $$(n+2)F_{n+1}(x) = 2(n+\lambda)xF_n(x) - nF_{n-1}(x)$$

and

(2.15) $$(1+x^2)F_n'(x) = n(F_{n-1}(x) - xF_n(x))$$

and let

$$\Delta_n^{(\lambda)}(x) = (F_n(x))^2 - F_{n-1}(x)F_{n+1}(x)$$

Employing (2.14) and (2.15) , and elementary calculation yields

$$\text{(2.16)} \qquad \frac{d}{dx}\{(1-x^2)^{\lambda-1}\Delta_n^{(\lambda)}(x)\} = -\frac{2\lambda}{n(n+2\lambda)}(1-x^2)^{\lambda-1}F_n(x)F_n'(x)$$

We shall prove the inequality $\Delta_n^{(\lambda)}(x) > 0$, $-1 < x < 1$, and $\lambda > 0$. We shall show in fact that the relative extrema of $(1-x^2)^{\lambda-1}\Delta_n^{(\lambda)}(x)$ are all positive. Observe at points x where this has a relative extremum, in view of (2.16) ,

$$F_n(x)F_n'(x) = 0 \quad .$$

If $F_n(x) = 0$ then $\Delta_n^{\lambda} = -F_{n-1}(x)F_{n+1}(x)$; furthermore from (2.14)

$$(n+2\lambda)F_{n+1}(x) = -nF_{n-1}(x) \ .$$

Hence

$$\Delta_n^{(\lambda)}(x) = n/n+2\,\lambda\ F_{n-1}^2(x) > 0$$

If on the other hand $F_n'(x) = 0$, then from (2.15)

$$F_{n-1}(x) = xF_n(x)$$

Hence from (2.15), we have

$$(n+2\lambda)F_{n+1}(x) = (2(n+2\lambda)-n)F_{n-1}(x) = (n+2\lambda)F_{n-1}(x)$$

It follows that

$$\Delta_n^{(\lambda)}(x) = (1-x^2)F_n^2(x) > 0 .$$

Hence the result.

(iii) The Bessel function.

He considers the new function as

$$\Lambda_\mu(t) = (2/t)^\mu \Gamma(\mu+1) J_\mu(t) = \Lambda_\mu(-t)$$

and establishes the recurrence formula as

$$(2.17) \qquad \frac{t^2 \Lambda_{\mu+1}(t)}{4\mu(\mu+1)} = \Lambda_\mu(t) - \Lambda_{\mu-1}(t) , \quad \mu > 0$$

and with the known result, we can establish the following result

$$(2.18) \qquad \Lambda_\mu'(t) = -t/2 \cdot \Lambda_{\mu+1}(t)/\mu+1 = 2\mu/t \cdot (\Lambda_{\mu-1}(t) - \Lambda_\mu(t))$$

We shall show that

$$\Delta_\mu(t) = (\Lambda_\mu(t))^2 - \Lambda_{\mu-1}(t)\Lambda_{\mu+1}(t) > 0 \ , \quad t > 0 \ \text{ and } \ \mu > 0$$

by showing the minima of $t^{2\mu+2}\Delta_\mu(t)$ for $t > 0$, are positive. We have

$$\Delta'_\mu(t) = 2\Lambda_\mu(t)\,\Lambda'_\mu(t) - \Lambda'_{\mu-1}(t)\Lambda_{\mu+1}(t) - \Lambda_{\mu-1}(t).\Lambda'_{\mu+1}(t)$$

and by a simple calculation and employing (2.17) , we have

$$\frac{d}{dt}\,(t^{2\mu+2}\Delta_\mu(t)) = -\frac{2}{\mu}.t^{2\mu+2}\ \Lambda_\mu(t)\Lambda'_\mu(t)$$

Thus the possible extrema of $t^{2\mu+2}\Delta_\mu(t)$ for $t > 0$ are reached when

$$\Lambda_\mu(t)\Lambda'_\mu(t) = 0$$

If $\Lambda_\mu(t) = 0$, then

$$\Lambda_\mu(t) = -\Lambda_{\mu-1}(t)\Lambda_{\mu+1}(t)$$

From (2.17)

$$t^2\Lambda_{\mu+1}(t) = -4\mu(\mu+1)\Lambda_{\mu-1}(t)$$

so that

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$$\Delta_\mu(t) = \frac{4\mu(\mu+1)}{t^2}\Lambda^2_{\mu-1}(t) > 0$$

If on the other hand, $\Lambda'_\mu(t) = 0$, then from (2.17) and (2.18) $\Lambda_{\mu-1}(t) = \Lambda_\mu(t)$ and $\Lambda_{\mu+1}(t) = 0$.

Thus $\Delta_\mu(t) = \Lambda^2_\mu(t) > 0$. Hence the inequality follows.

CHAPTER III

DIFFERENTIAL EQUATIONS SATISFIED BY CERTAIN TURÁN EXPRESSIONS

In this chapter we are going to consider some differential equations satisfied by the Turán expressions for the classical orthogonal polynomials.

For convenience, we shall adopt the following notation.

$$F_n(x) = \Delta_n(f)$$

Theorem 11. The polynomial $H_n(x)$ satisfies the differential equation

$$H_n'''(x) - 6xH_n''(x) + 2(4x^2+4n-3)H_n'(x) - 16x(n-1)H_n(x) = 0$$

where $H_n(x)$ is the Turán expression for the Hermite polynomials with the following initial conditions

$$H_n(0) = \begin{cases} 2^{2n}[(\frac{1}{2})_{n/2}]^2, & n \text{ even} \\ 2^{2n}(\frac{1}{2})_{\frac{n+1}{2}}(\frac{1}{2})_{\frac{n-1}{2}}, & n \text{ odd} \end{cases} \qquad H_n'(0) = 0$$

$$\mathcal{H}''_n(0) = \begin{cases} 0 & n \text{ odd} \\ (-1)^n 2^{2n+2}[(\tfrac{1}{2})_{n/2}]^2 & n \text{ even} \end{cases} \qquad \mathcal{H}'''_n(0) = 0$$

Proof: We know that the Hermite polynomials satisfy the following relations:

(3.1) $$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

(3.2) $$H'_n(x) = 2nH_{n-1}(x)$$

(3.3) $$H_{n+1}(x) = 2xH_n(x) - H'_n(x)$$

(3.4) $$\mathcal{H}_n(x) = 2(n-1)\mathcal{H}_{n-1}(x) + 2H^2_{n-1}(x)$$

(3.5) $$\delta_n(H) = H_{n+1}(x)H_{n+2}(x) - H_n(x)H_{n+3}(x)$$

Differentiating $\mathcal{H}_n(x)$ with respect to x, we get

$$\mathcal{H}'_n(x) = 2H_n(x)H'_n(x) - H'_{n-1}(x)H_{n+1}(x) - H_{n-1}(x)H'_{n+1}(x)$$

$$= 2(n-1)[H_{n-1}(x)H_n(x)-H_{n-2}(x)H_{n+1}(x)]$$

or

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$$\mathcal{H}'_n(x) = 2(n-1)\delta_{n-2}(H) \tag{3.6}$$

This result has been proved by Toscano [54] . Again, applying (3.3) , we have

$$\mathcal{H}'_n(x) = 2H_n(x)[2xH_n(x)-H_{n+1}(x)] - H_{n+1}(x)[2xH_{n-1}(x)-H_n(x)]$$

$$- H_{n-1}(x)[2xH_{n+1}(x)-H_{n+2}(x)]$$

$$= 4x\mathcal{H}_n(x) - H_n(x)H_{n+1}(x) + H_{n-1}(x)H_{n+2}(x)$$

$$= 4x\mathcal{H}_n(x) - H_n(x)[2xH_n(x)-2nH_{n-1}(x)]$$

$$+ H_{n-1}(x)[2xH_{n+1}(x)-2(n+1)H_n(x)]$$

or

$$\mathcal{H}'_n(x) = 2x\mathcal{H}_n(x) - 2H_n(x)H_{n-1}(x) \tag{3.7}$$

Differentiating (3.7) , we get

$$\mathcal{H}''_n(x) = 2x\mathcal{H}'_n(x) + 2\mathcal{H}_n(x) - 2H_n(x)H'_{n-1}(x) - 2H'_n(x)H_{n-1}(x)$$

$$= 2x\mathcal{H}'_n(x) + 2\mathcal{H}_n(x) - 4(n-1)H_n(x)H_{n-2}(x) - 4nH^2_{n-1}(x)$$

$$= 2x\mathcal{H}'_n(x) + 2\mathcal{H}_n(x) - 4(n-1)[H^2_{n-1}(x)-\mathcal{H}_{n-1}(x)] - 4nH^2_{n-1}(x)$$

or

$$H''_n(x) = 2xH'_n(x) + 2H_n(x) + 2H_n(x) - 4H^2_{n-1}(x) - 4(2n-1)H^2_{n-1}(x)$$

(3.8) $$H''_n(x) = 2xH'_n(x) + 4H_n(x) - 8nH^2_{n-1}(x)$$

Differentiating (3.8) , we have

$$H'''_n(x) = 2xH''_n(x) + 2H'_n(x) + 4H'_n(x) - 16nH_{n-1}(x)H'_{n-1}(x)$$

$$= 2xH''_n(x) + 6H'_n(x) - 16nH_{n-1}(x)[2xH_{n-1}(x)-H_n(x)]$$

or

$$H'''_n(x) = 2xH''_n(x) + 6H'_n(x) - 32nxH^2_{n-1}(x) + 16nH_n(x)H_{n-1}(x)$$

$$= 2xH''_n(x) + 6H'_n(x) - 32nx[\frac{2xH'_n(x)+4H_n(x)-H''_n(x)}{8n}]$$

$$+ 16n[\frac{2xH_n(x)-H'_n(x)}{2}]$$

or

$$H'''_n(x) = 2xH''_n(x) + 6H'_n(x) - 8x^2H'_n(x) - 16xH_n(x) + 4xH''_n(x) + 16nxH_n(x) - 8nH'_n(x)$$

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or

$$H_n'''(x) - 6xH_n''(x) + 2(4x^2+4n-3)H_n'(x) - 16x(n-1)H_n(x) = 0$$

which is the required result.

Theorem 12. The polynomial $L_n^{(\alpha)}(x)$ satisfies the differential equation

$$x^2 L_n^{(\alpha)'''}(x) + [3(\alpha-x)+1]xL_n^{(\alpha)''}(x) + [4nx-3\alpha-3+4x+2(\alpha+1-x)^2]L_n^{(\alpha)'}(x)$$

$$+ 2n[2(\alpha-x)-1]L_n^{(\alpha)}(x) = 0$$

where $L_n^{(\alpha)}(x)$ is the Turán expression for the general Laguerre polynomial with the following initial conditions

$$L_n^{(\alpha)}(0) = \frac{\alpha(1+\alpha)_n(1+\alpha)_{n-1}}{n!(n+1)!} ,$$

$$L_n^{(\alpha)'}(0) = \frac{-2\alpha(1+\alpha)_n(2+\alpha)_{n-2}}{(n-1)!(n+1)!}$$

$$L_n^{(\alpha)''}(0) = \frac{2(1+\alpha)_n(2+\alpha)_{n-2}}{(n-1)!n!} - \frac{(3+\alpha)_{n-2}(1+\alpha)_{n-1}}{n!(n-1)!}(1+2n+\alpha) + \frac{(3+\alpha)_{n-3}(1+\alpha)_n}{(n-2)!(n+1)!} \times$$

$$\times (2n+3\alpha+2)$$

$$L_n^{(\alpha)'''}(0) = -\frac{6(2+\alpha)_{n-1}(3+\alpha)_{n-2}}{(n-1)!(n-2)!} - \frac{2(1+\alpha)_n(4+\alpha)_{n-3}}{n!(n-3)!} + \frac{3(2+\alpha)_{n-2}(3+\alpha)_{n-1}}{(n-2)!(n-1)!}$$

$$+ \frac{3(3+\alpha)_{n-3}(2+\alpha)_n}{(n-3)n!} + \frac{(1+\alpha)_n(\alpha+4)_{n-4}}{n!(n-4)!} + \frac{(1+\alpha)_{n-1}(4+\alpha)_{n-2}}{(n-1)!(n-2)!}$$

Proof: The general Laguerre polynomials satisfy the following relations

$$(n+1)L_{n+1}^{(\alpha)}(x) + (x-\alpha-2n+1)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0 \tag{3.9}$$

$$\frac{d}{dx} L_n^{(\alpha)}(x) - \frac{d}{dx} L_{n-1}^{(\alpha)}(x) = -L_{n-1}^{(\alpha)}(x) \tag{3.10}$$

$$x \frac{d^2 L_n^{(\alpha)}(x)}{dx^2} + (\alpha+1-x)\frac{dL_n^{(\alpha)}(x)}{dx} + nL_n^{(\alpha)}(x) = 0 \tag{3.11}$$

$$x \frac{dL_n^{(\alpha)}(x)}{dx} = nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x) \tag{3.12}$$

Differentiating (3.12) and subtracting from (3.11) , we have

$$(n+\alpha)\frac{dL_{n-1}^{(\alpha)}(x)}{dx} - (\alpha-x+n)\frac{dL_n^{(\alpha)}(x)}{dx} = nL_n^{(\alpha)}(x) \tag{3.13}$$

$\mathcal{L}_n^{(\alpha)}(x)$ and $D_n(L^{(\alpha)})$ are defined as

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$$\mathcal{L}_n^{(\alpha)}(x) = L_n^{(\alpha)^2}(x) - L_{n+1}^{(\alpha)}(x)L_{n-1}^{(\alpha)}(x) \tag{3.14}$$

$$D_n(L^{(\alpha)}) = L_n^{(\alpha)'^2}(x) - L_{n+1}^{(\alpha)'}(x)L_{n-1}^{(\alpha)'}(x) \quad . \tag{3.15}$$

Denote

$$\begin{cases} A_n = L_n^{(\alpha)}(x)L_n^{(\alpha)'}(x) - L_{n-1}^{(\alpha)}(x)L_{n+1}^{(\alpha)'}(x) \\ \\ B_n = L_{n-1}^{(\alpha)'}(x)L_{n+1}^{(\alpha)}(x) - L_n^{(\alpha)}(x)L_n^{(\alpha)'}(x) \end{cases} \tag{3.16}$$

Differentiating (3.14) we have

$$\mathcal{L}_n^{(\alpha)'}(x) = 2L_n^{(\alpha)}(x)L_n^{(\alpha)'}(x) - L_{n-1}^{(\alpha)}(x)L_{n+1}^{(\alpha)'}(x) - L_{n-1}^{(\alpha)'}(x)L_{n+1}^{(\alpha)}(x)$$

Differentiating, again, we get

$$\begin{aligned} \mathcal{L}_n^{(\alpha)''}(x) &= 2L_n^{(\alpha)'^2}(x) + 2L_n^{(\alpha)}(x)L_n^{(\alpha)''}(x) - 2L_{n-1}^{(\alpha)'}(x)L_{n+1}^{(\alpha)'}(x) \\ &\quad - L_{n+1}^{(\alpha)}(x)L_{n-1}^{(\alpha)''}(x) - L_{n-1}^{(\alpha)}(x)L_{n+1}^{(\alpha)''}(x) \\ &= 2[L_n^{(\alpha)'^2}(x) - L_{n-1}^{(\alpha)'}(x)L_{n+1}^{(\alpha)'}(x)] + 2\,\frac{L_n^{(\alpha)}(x)}{x}\,[-(\alpha+1-x)L_n^{(\alpha)'}(x) \\ &\quad - nL_n^{(\alpha)}(x)] - \frac{L_{n+1}^{(\alpha)}(x)}{x}[-(\alpha+1-x)L_{n-1}^{(\alpha)'}(x) - (n-1)L_{n-1}^{(\alpha)}(x)] \end{aligned}$$

$$- \frac{L_{n-1}^{(\alpha)}(x)}{x} [-(\alpha+1-x)L_{n+1}^{(\alpha)'}(x)-(n+1)L_{n+1}^{(\alpha)}(x)]$$

or

$$(3.17) \qquad L_n^{(\alpha)''}(x) + \frac{(\alpha+1-x)}{x} L_n^{(\alpha)'}(x) + \frac{2n}{x} L_n^{(\alpha)}(x) = 2D_n(L^{(\alpha)})$$

Differentiating (3.17) , we have

$$L_n^{(\alpha)'''}(x) - \frac{(\alpha+1)}{x^2} L_n^{(\alpha)'}(x) + (\frac{\alpha+1}{x} -1) L_n^{(\alpha)''}(x) - \frac{2n}{x^2} L_n^{(\alpha)}(x)$$

$$+ \frac{2n}{x} L_n^{(\alpha)'}(x) = 2D_n'(L^{(\alpha)})$$

or

$$L_n^{(\alpha)'''}(x) + (\frac{\alpha+1}{x} -1) L_n^{(\alpha)''}(x) - (\frac{\alpha+1}{x^2} - \frac{2n}{x}) L_n^{(\alpha)'}(x) - \frac{2n}{x^2} L_n^{(\alpha)}(x)$$

$$= 2[2L_n^{(\alpha)'}(x)L_n^{(\alpha)''}(x)-L_{n-1}^{(\alpha)'}(x)L_{n+1}^{(\alpha)''}(x)-L_{n-1}^{(\alpha)''}(x)L_{n+1}^{(\alpha)'}(x)]$$

$$= \frac{2}{x} [2L_n^{(\alpha)'}(x)\{(x-1-\alpha)L_n^{(\alpha)'}(x)-nL_n^{(\alpha)}(x)\}$$

$$- L_{n-1}^{(\alpha)'}(x)\{(x-1-\alpha)L_{n+1}^{(\alpha)'}(x)-(n+1)L_{n+1}^{(\alpha)}(x)\}$$

$$- L_{n+1}^{(\alpha)'}(x)\{(x-1-\alpha)L_{n-1}^{(\alpha)'}(x)-(n-1)L_{n-1}^{(\alpha)}(x)\}]$$

$$x^2 L_n^{(\alpha)'''}(x) + [(\alpha+1)x-x^2]L_n^{(\alpha)''}(x) + (2nx-\alpha-1)L_n^{(\alpha)'}(x)$$

$$- 2nL_n^{(\alpha)}(x) = 2x[2(x-1-\alpha)(L_n^{(\alpha)'2}(x)-L_{n-1}^{(\alpha)'}(x)L_{n+1}^{(\alpha)'}(x))$$

$$- n(2L_n^{(\alpha)}(x)L_n^{(\alpha)'}(x)-L_{n-1}^{(\alpha)'}(x)L_{n+1}^{(\alpha)}(x)-L_{n-1}^{(\alpha)}(x)L_{n+1}^{(\alpha)'}(x))$$

$$+ L_{n+1}^{(\alpha)}(x)L_{n-1}^{(\alpha)'}(x)-L_{n-1}^{(\alpha)}(x)L_{n+1}^{(\alpha)'}(x)]$$

or

$$x^2 L_n^{(\alpha)'''}(x) + ((\alpha+1)x-x^2)L_n^{(\alpha)''}(x)+(2nx-\alpha-1)L_n^{(\alpha)'}(x)$$

$$- 2nL_n^{(\alpha)}(x) = 2x[(x-1-\alpha)\{L_n^{(\alpha)''}(x)+(\frac{\alpha+1-x}{x})L_n^{(\alpha)'}(x)$$

$$+ \frac{2n}{x} L_n^{(\alpha)}(x)\}-nL_n^{(\alpha)'}(x)+(A_n+B_n)]$$

or

$$x^2 L_n^{(\alpha)'''}(x) + [(\alpha+1)x-x^2-2x^2+2(\alpha+1)x]L_n^{(\alpha)''}(x)$$

$$+ [(2nx-\alpha-1)+2(\alpha+1-x)^2+2nx]L_n^{(\alpha)'}(x)$$

$$+ [-2n-4n(x-1-\alpha)]L_n^{(\alpha)}(x) = 2x(A_n+B_n)$$

or

$$x^2 L_n^{(\alpha)'''}(x) + 3x(\alpha-x+1)L_n^{(\alpha)''}(x) + [2nx-\alpha-1+2(\alpha+1-x)^2+2nx] \times$$

(3.18)

$$\times\ L_n^{(\alpha)'}(x) + 2n(2(\alpha+1-x)-1)L_n^{(\alpha)}(x) = 2x(A_n+B_n)$$

Now we shall find the value of (A_n+B_n)

$$L_n^{(\alpha)}(x) = L_n^{(\alpha)'}(x) - L_{n+1}^{(\alpha)'}(x)$$

Multiplying by $L_n^{(\alpha)'}(x)$, we get

$$L_n^{(\alpha)}(x)L_n^{(\alpha)'}(x) = L_n^{(\alpha)'^2}(x) - L_{n+1}^{(\alpha)'}(x)L_n^{(\alpha)'}(x) \tag{3.19}$$

Similarly, we have

$$L_{n-1}^{(\alpha)}(x)L_{n+1}^{(\alpha)'}(x) = L_{n-1}^{(\alpha)'}(x)L_{n+1}^{(\alpha)'}(x) - L_n^{(\alpha)'}(x)L_{n+1}^{(\alpha)'}(x) \tag{3.20}$$

Subtracting (3.20) from (3.19) , we get

(3.21)

$$A_n = L_n^{(\alpha)'^2}(x) - L_{n-1}^{(\alpha)'}(x)L_{n+1}^{(\alpha)'}(x)$$

$$= D_n(L^{(\alpha)})$$

Again, we have

$$L_{n+1}^{(\alpha)}(x)L_{n-1}^{(\alpha)'}(x) = L_{n+1}^{(\alpha)'}(x)L_{n-1}^{(\alpha)'}(x) - L_{n-1}^{(\alpha)'}(x)L_{n+2}^{(\alpha)'}(x) \tag{3.22}$$

Subtracting (3.19) from (3.22) , we get

$$B_n = [L_{n+1}^{(\alpha)'}(x)L_n^{(\alpha)'}(x) - L_{n-1}^{(\alpha)'}(x)L_{n+2}^{(\alpha)'}(x)] - D_n(L^{(\alpha)})$$

Therefore

$$A_n + B_n = L_{n+1}^{(\alpha)'}(x)L_n^{(\alpha)'}(x) - L_{n-1}^{(\alpha)'}(x)L_{n+2}^{(\alpha)'}(x) \tag{3.23}$$

But

$$\mathcal{L}_n^{(\alpha)'}(x) = 2L_n^{(\alpha)}(x)L_n^{(\alpha)'}(x) - L_{n-1}^{(\alpha)}(x)L_{n+1}^{(\alpha)'}(x) - L_{n-1}^{(\alpha)'}(x)L_{n+1}^{(\alpha)}(x)$$

or

$$\mathcal{L}_n^{(\alpha)'}(x) = A_n - B_n \tag{3.24}$$

Adding (3.23) and (3.24) , we get

$$2A_n = \mathcal{L}_n^{(\alpha)'}(x) + [L_{n+1}^{(\alpha)'}(x)L_n^{(\alpha)'}(x) - L_{n-1}^{(\alpha)'}(x)L_{n+2}^{(\alpha)'}(x)]$$

or

$$2D_n(L^{(\alpha)}) - L_n^{(\alpha)'}(x) = L_{n+1}^{(\alpha)'}(x)L_n^{(\alpha)'}(x) - L_{n-1}^{(\alpha)'}(x)L_{n+2}^{(\alpha)'}(x)$$

$$= A_n + B_n$$

Substituting in (3.18) , we have

$$x^2L_n^{(\alpha)'''}(x) + 3x(\alpha-x+1)L_n^{(\alpha)''}(x) + [4nx-\alpha-1+(\alpha+1-x)^2 2]L_n^{(\alpha)'}(x)$$

$$+ 2n[2(\alpha+1-x)-1]L_n^{(\alpha)}(x) = 2x[2D_n(L^{(\alpha)})-L_n^{(\alpha)'}(x)]$$

or

$$x^2L_n^{(\alpha)'''}(x) + 3x(\alpha-x+1)L_n^{(\alpha)''}(x) + [4nx-\alpha-1+2(\alpha+1-x)^2+2x]L_n^{(\alpha)'}(x)$$

$$+ 2n[2(\alpha+1-x)-1]L_n^{(\alpha)}(x) = 2x[L_n^{(\alpha)''}(x)+(\frac{\alpha+1-x}{x})L_n^{(\alpha)'}(x)$$

$$+ \frac{2n}{x} L_n^{(\alpha)}(x)]$$

or

$$x^2L_n^{(\alpha)'''}(x) + [3(\alpha-x)+1]xL_n^{(\alpha)''}(x)+[4nx-3\alpha-3+4x+2(\alpha+1-x)^2]L_n^{(\alpha)'}(x)$$

$$+ 2n[2(\alpha-x)-1]L_n^{(\alpha)}(x) = 0 \quad .$$

<u>Theorem 13.</u> <u>The polynomial</u> $\mathcal{P}_n^{(\lambda)}(x)$ <u>satisfies the differential equation</u>

$$(1-x^2)^2\mathcal{P}_n^{(\lambda)'''}(x) - x(1-x^2)(1+6\lambda)\mathcal{P}_n^{(\lambda)''}(x) + [\{4n(n+2\lambda)+2\lambda-1\}(1-x^2)$$

$$- 2x^2(1-4\lambda^2)]\mathcal{P}_n^{(\lambda)'}(x) + 8nx(n+2\lambda)(1-\lambda)\mathcal{P}_n^{(\lambda)}(x) = 0$$

<u>where</u> $\mathcal{P}_n^{(\lambda)}(x)$ <u>is the Turán expression for the Ultraspherical polynomial with the initial conditions</u>

$$\mathcal{P}_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}^2 - \binom{n+2\lambda}{n+1}\binom{n+2\lambda-2}{n-1},$$

$$\mathcal{P}_n^{(\lambda)'}(1) = \frac{2\lambda(1-\lambda)}{n+\lambda}\left[\binom{n+2\lambda}{n+1}\binom{n+2\lambda-1}{n-2}-\binom{n+2\lambda-2}{n-1}\binom{n+2\lambda+1}{n}\right.$$

$$\mathcal{P}_n^{(\lambda)''}(1) = \frac{4\lambda(1-\lambda^2)}{n+\lambda}\left[\binom{n+2\lambda}{n+1}\binom{n+2\lambda}{n-3}-\binom{n+2\lambda-2}{n-1}\binom{n+2\lambda+2}{n-1}\right]$$

$$\mathcal{P}_n^{(\lambda)'''}(1) = \frac{8\lambda(1-\lambda^2)}{n+\lambda}\left[\binom{n+2\lambda}{n+1}\binom{n+2\lambda+1}{n-4}(\lambda+2)-\binom{n+2\lambda-2}{n-1}\binom{n+2\lambda+3}{n-2}(\lambda+2)\right.$$

$$\left.+\lambda\binom{n+2\lambda+1}{n}\binom{n+2\lambda}{n-3}-\lambda\binom{n+2\lambda-1}{n-2}\binom{n+2\lambda+2}{n-1}\right]$$

Proof: The Ultraspherical polynomials satisfy the following relations

$$(3.25) \qquad (1-x^2)\frac{d^2P_n^{(\lambda)}(x)}{dx^2} - x(2\lambda+1)\frac{dP_n^{(\lambda)}(x)}{dx} + n(n+2\lambda)P_n^{(\lambda)}(x) = 0$$

$$nP_n^{(\lambda)}(x) = x\frac{d}{dx}P_n^{(\lambda)}(x) - \frac{d}{dx}P_{n-1}^{(\lambda)}(x) \tag{3.26}$$

$$(n+2\lambda)P_n^{(\lambda)}(x) = \frac{d}{dx}P_{n+1}^{(\lambda)}(x) - x\frac{d}{dx}P_n^{(\lambda)}(x) \tag{3.27}$$

$$2(n+\lambda)P_n^{(\lambda)}(x) = \frac{d}{dx}P_{n+1}^{(\lambda)}(x) - \frac{d}{dx}P_{n-1}^{(\lambda)}(x) \tag{3.28}$$

We shall establish the following relation (3.29) for its use in finding out the differential equation for the polynomial $P_n^{(\lambda)}(x)$ which is the Turán expression for the Ultraspherical polynomial.

Consider

$$n(P_n^{(\lambda)}(x)P_n^{(\lambda)'}(x)-P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x))$$

$$- (n+2\lambda)(P_{n+1}^{(\lambda)}(x)-P_n^{(\lambda)}(x)P_n^{(\lambda)'}(x))$$

$$= 2(n+\lambda)P_n^{(\lambda)}(x)P_n^{(\lambda)'}(x) - (n+2\lambda-1)P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x)$$

$$- (n+1)P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)'}(x) - (2\lambda-1)[P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)'}(x)$$

$$- P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x)]$$

From (3.26), (3.27) and (3.28) , we have

$$2(n+\lambda)P_n^{(\lambda)}(x)P_n^{(\lambda)'}(x) - (n+2\lambda-1)P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x) - (n+1)P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)}(x) = 0$$

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or

$$(3.29) \qquad 2P_n^{(\lambda)}P_n^{(\lambda)'}(x) = \frac{(n+2\lambda-1)}{n+\lambda} P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x) + \frac{n+1}{n+\lambda} P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)'}(x)$$

Let

$$\mathcal{P}_n^{(\lambda)}(x) = P_n^{(\lambda)^2}(x) - P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)}(x)$$

Differentiating with respect to x , we have

$$\mathcal{P}_n^{(\lambda)'}(x) = 2P_n^{(\lambda)}(x)P_n^{(\lambda)'}(x) - P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x)$$

$$- P_{n-1}^{(\lambda)'}(x)P_{n+1}^{(\lambda)}(x)$$

$$= \frac{(n+2\lambda-1)}{n+\lambda} P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x) + \frac{n+1}{n+\lambda} P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)'}(x)$$

$$- P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x) - P_{n-1}^{(\lambda)'}(x)P_{n+1}^{(\lambda)}(x)$$

$$(3.30) \qquad = \frac{\lambda-1}{n+\lambda} P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x) + \frac{1-\lambda}{n+\lambda} P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)'}(x)$$

Again, differentiating, we get

$$\mathcal{P}_n^{(\lambda)''}(x) = \frac{1-\lambda}{n+\lambda} [P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)''}(x) - P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)''}(x)]$$

or , using (3.25) , we get

$$(1-x^2)P_n^{(\lambda)''}(x) = \frac{1-\lambda}{n+\lambda}[P_{n+1}^{(\lambda)}(x)\{x(2\lambda+1)P_{n-1}^{(\lambda)'}(x)-(n-1)(n+2\lambda-1)P_{n-1}^{(\lambda)}(x)\}$$

$$- P_{n-1}^{(\lambda)}(x)\{x(2\lambda+1)P_{n+1}^{(\lambda)'}(x)-(n+1)(n+2\lambda+1)P_{n+1}^{(\lambda)}(x)\}]$$

or

$$(1-x^2)P_n^{(\lambda)''}(x) = \frac{1-\lambda}{n+\lambda}\, x(2\lambda+1)(P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)'}(x)-P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x))$$

$$+ 4(1-\lambda)P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)}(x)$$

or

$$(3.31) \qquad (1-x^2)P_n^{(\lambda)''}(x) - x(2\lambda+1)P_n^{(\lambda)'}(x) = 4(1-\lambda)P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)}(x)$$

Again, differentiating, we have

$$(3.32) \qquad \begin{aligned}&(1-x^2)P_n^{(\lambda)'''}(x) - x(2\lambda+3)P_n^{(\lambda)''}(x) - (2\lambda+1)P_n^{(\lambda)'}(x)\\ &= 4(1-\lambda)[P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)'}(x)+P_{n+1}^{(\lambda)'}(x)P_{n-1}^{(\lambda)}(x)]\end{aligned}$$

Starting, again, from the Turán expression $P_n^{(\lambda)}(x)$. Differentiating it, we have

$$P_n^{(\lambda)'}(x) = 2P_n^{(\lambda)}(x)P_n^{(\lambda)'}(x) - P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x) - P_{n-1}^{(\lambda)'}(x)P_{n+1}^{(\lambda)}(x)$$

Again, differentiating, we have

$$P_n^{(\lambda)''}(x) = 2[P_n^{(\lambda)'^2}(x)-P_{n-1}^{(\lambda)'}(x)P_{n+1}^{(\lambda)'}(x)] + 2P_n^{(\lambda)}(x)P_n^{(\lambda)''}(x)$$

$$- P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)''}(x) - P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)''}(x)$$

or, using (3.25), we get

$$(1-x^2)P_n^{(\lambda)''}(x) = 2(1-x^2)D_n(P^{(\lambda)}) + 2P_n^{(\lambda)}(x)[x(2\lambda+1)P_n^{(\lambda)'}(x)-n(n+2\lambda)P_n^{(\lambda)}(x)]$$

$$- P_{n-1}^{(\lambda)}(x)[x(2\lambda+1)P_{n+1}^{(\lambda)'}(x)-(n+1)(n+2\lambda+1)P_{n+1}^{(\lambda)}(x)]$$

$$- P_{n+1}^{(\lambda)}(x)[x(2\lambda+1)P_{n-1}^{(\lambda)'}(x)-(n-1)(n+2\lambda-1)P_{n-1}^{(\lambda)}(x)]$$

where

$$D_n(P^{(\lambda)}) = P_n^{(\lambda)'^2}(x) - P_{n+1}^{(\lambda)'}(x)P_{n-1}^{(\lambda)'}(x)$$

or

$$(1-x^2)P_n^{(\lambda)''}(x) = 2(1-x^2)D_n(P^{(\lambda)}) + x(2\lambda+1)P_n^{(\lambda)'}(x)$$

$$- 2n(n+2\lambda)P_n^{(\lambda)^2}(x) + (n+1)(n+2\lambda+1)P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)}(x)$$

$$+ (n-1)(n+2\lambda-1)P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)}(x)$$

$$= 2(1-x^2)D_n(P^{(\lambda)}) + x(2\lambda+1)P_n^{(\lambda)'}(x)$$

$$- 2n(n+2\lambda)P_n^{(\lambda)^2}(x) + (n+1)(n+2\lambda)P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)}(x)$$

$$+ (n-1)(n+2\lambda)P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)}(x) + 2P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)}(x)$$

or

$$(3.33)\quad \begin{aligned} &(1-x^2)P_n^{(\lambda)''}(x) - x(2\lambda+1)P_n^{(\lambda)'}(x) + 2n(n+2\lambda)P_n^{(\lambda)}(x) \\ &\qquad = 2(1-x^2)D_n(P^{(\lambda)}) + 2P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)}(x) \end{aligned}$$

Multiplying (3.33) by $2(1-\lambda)$, we get

$$(3.34)\quad \begin{aligned} &2(1-\lambda)(1-x^2)P_n^{(\lambda)''}(x) - 2x(2\lambda+1)(1-\lambda)P_n^{(\lambda)'}(x) + 4n(n+2\lambda)(1-\lambda)P_n^{(\lambda)}(x) \\ &\qquad = 4(1-x^2)(1-\lambda)D_n(P^{(\lambda)}) + 4(1-\lambda)P_{n+1}^{(\lambda)}(x)P_{n-1}^{(\lambda)}(x) \end{aligned}$$

Subtracting (3.31) from (3.34), we have

$$(3.35)\quad \begin{aligned} &(1-2\lambda)(1-x^2)P_n^{(\lambda)''}(x) - x(2\lambda+1)(1-2\lambda)P_n^{(\lambda)'}(x) \\ &\qquad + 4n(n+2\lambda)(1-\lambda)P_n^{(\lambda)}(x) = 4(1-x^2)(1-\lambda)D_n(P^{(\lambda)}) \end{aligned}$$

Differentiating again, we get

$$(1-x^2)(1-2\lambda)P_n^{(\lambda)'''}(x) - 2x(1-2\lambda)P_n^{(\lambda)''}(x) - x(2\lambda+1)(1-2\lambda)P_n^{(\lambda)''}(x)$$

$$- (1+2\lambda)(1-2\lambda)P_n^{(\lambda)'}(x) + 4n(n+2\lambda)(1-\lambda)P_n^{(\lambda)'}(x)$$

$$= -8x(1-\lambda)D_n(P^{(\lambda)}) + 4(1-x^2)(1-\lambda)[2P_n^{(\lambda)'}(x)P_n^{(\lambda)''}(x)$$

$$- P_{n-1}^{(\lambda)'}(x)P_{n+1}^{(\lambda)''}(x)-P_{n+1}^{(\lambda)'}(x)P_{n-1}^{(\lambda)''}(x)]$$

or

$$(1-x^2)(1-2\lambda)P_n^{(\lambda)'''}(x) - x(1-2\lambda)(2\lambda+3)P_n^{(\lambda)''}(x)$$

$$+ [4n(n+2\lambda)(1-\lambda)-(1-4\lambda^2)]P_n^{(\lambda)'}(x)$$

$$= -8x(1-\lambda)D_n(P^{(\lambda)}) + 4(1-\lambda)[2P_n^{(\lambda)'}(x)\{x(2\lambda+1)P_n^{(\lambda)'}(x)$$

$$- n(n+2\lambda)P_n^{(\lambda)}(x)\} - P_{n-1}^{(\lambda)'}(x)\{x(2\lambda+1)P_{n+1}^{(\lambda)'}(x)$$

$$- (n+1)(n+2\lambda+1)P_{n+1}^{(\lambda)}(x)\} - P_{n+1}^{(\lambda)'}(x)\{x(2\lambda+1)$$

$$\times P_{n-1}^{(\lambda)'}(x) - (n-1)(n+2\lambda-1)P_{n-1}^{(\lambda)}(x)\}]$$

$$= -8x(1-\lambda)D_n(P^{(\lambda)}) + 8(1-\lambda)x(2\lambda+1)D_n(P^{(\lambda)})$$

$$- 4n(1-\lambda)(n+2\lambda)(2P_n^{(\lambda)}(x)P_n^{(\lambda)'}(x))$$

$$+ 4n(n+2\lambda)(1-\lambda)P_{n-1}^{(\lambda)'}(x)P_{n+1}^{(\lambda)}(x)$$

$$+ 4n(n+2\lambda)(1-\lambda)P_{n-1}^{(\lambda)}(x)P_{n+1}^{(\lambda)'}(x)$$

$$+ P_{n-1}^{(\lambda)'}(x)P_{n+1}^{(\lambda)}(x)[(2n+2\lambda+1)4(1-\lambda)]$$

$$+ P_{n+1}^{(\lambda)'}(x)P_{n-1}^{(\lambda)}(x)[-n-n+1-2\lambda][4(1-\lambda)]$$

$$= 16x\lambda(1-\lambda)D_n(P^{(\lambda)}) - 4n(n+2\lambda)(1-\lambda)P_n^{(\lambda)'}(x)$$

$$+ 8(1-\lambda)(n+\lambda)[P_{n-1}^{(\lambda)'}(x)P_{n+1}^{(\lambda)}(x)-P_{n+1}^{(\lambda)'}(x)P_{n-1}^{(\lambda)}(x)]$$

$$+ 4(1-\lambda)[P_{n-1}^{(\lambda)'}(x)P_{n+1}^{(\lambda)}(x)+P_{n+1}^{(\lambda)'}(x)P_{n-1}^{(\lambda)}(x)]$$

or , using (3.30) and (3.32) , we get

$$(1-x^2)(1-2\lambda)P_n^{(\lambda)'''}(x) - x(1-2\lambda)(2\lambda+3)P_n^{(\lambda)''}(x)$$

$$+ [4n(n+2\lambda)(1-\lambda)-(1-4\lambda^2)]P_n^{(\lambda)'}(x)$$

$$= 16x\lambda(1-\lambda)D_n(P^{(\lambda)}) - 4n(n+2\lambda)(1-\lambda)P_n^{(\lambda)'}(x) + (1-x^2)P_n^{(\lambda)'''}(x)$$

$$- x(2\lambda+3)P_n^{(\lambda)''}(x) - (2\lambda+1)P_n^{(\lambda)'}(x)$$

$$+ 8(1-\lambda)(n+\lambda)\frac{(n+\lambda)}{(1-\lambda)} P_n^{(\lambda)'}(x)$$

or , using (3.35) , we get

$$-2\lambda(1-x^2)P_n^{(\lambda)'''}(x) + x(2\lambda)(2\lambda+3)P_n^{(\lambda)''}(x)$$

$$+ [8n(n+2\lambda)(1-\lambda)-(1-4\lambda^2)+(2\lambda+1)-8(n+\lambda)^2]P_n^{(\lambda)'}(x)$$

$$= \frac{4\lambda x}{1-x^2}[(1-x^2)(1-2\lambda)P_n^{(\lambda)''}(x)-x(2\lambda+1)(1-2\lambda)P_n^{(\lambda)'}(x)$$

$$+ 4n(n+2\lambda)(1-\lambda)P_n^{(\lambda)}(x)]$$

or

$$-2\lambda(1-x^2)P_n^{(\lambda)'''}(x) + 2\lambda x(2\lambda+3)(1-x^2)P_n^{(\lambda)''}(x) - 4\lambda x(1-x^2)(1-2\lambda)P_n^{(\lambda)''}(x)$$

$$+ [\{8n(n+2\lambda)(1-\lambda)-(1-4\lambda^2)-8(n+\lambda)^2+(2\lambda+1)\}(1-x^2)$$

$$+ 4\lambda x^2(2\lambda+1)(1-2\lambda)]P_n^{(\lambda)'}(x)$$

$$- 16\lambda xn(n+2\lambda)(1-\lambda)P_n^{(\lambda)}(x) = 0$$

or

$$(1-x^2)^2 P_n^{(\lambda)'''}(x) - x(1-x^2)(1+6\lambda)P_n^{(\lambda)''}(x)$$

$$+ [\{4n(n+2\lambda)+2\lambda-1\}(1-x^2)-2x^2(1-4\lambda^2)]\mathcal{P}_n^{(\lambda)'}(x) + 8nx(n+2\lambda)(1-\lambda)\mathcal{P}_n^{(\lambda)}(x) = 0$$

Corollary. Let $\lambda = \frac{1}{2}$, the Ultraspherical polynomial is reduced to the Legendre polynomial and the differential equation which is satisfied by the Turán expression for the Legendre polynomial, $\mathcal{P}_n(x)$, becomes

$$(1-x^2)^2\mathcal{P}_n'''(x) - 4x(1-x^2)\mathcal{P}_n''(x) + 4n(n+1)(1-x^2)\mathcal{P}_n'(x)$$

$$+ 4nx(n+1)\mathcal{P}_n(x) = 0$$

where

$$\mathcal{P}_n(x) = P_n^2(x) - P_{n+1}(x)P_{n-1}(x)$$

with the following initial conditions

$$\mathcal{P}_n(1) = 0$$

$$\mathcal{P}_n'(1) = -1$$

$$\mathcal{P}_n''(1) = -\frac{3n^2(n+1)}{4(2n+1)}$$

$$\mathcal{P}_n'''(1) = -\frac{n(n-1)(n+1)(n+2)(9n+2)}{16(2n+1)}$$

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CHAPTER IV

MISCELLANEOUS RESULTS CONCERNING TURÁN INEQUALITY

The Turán inequality for certain polynomials

1. Many relations involving finite series of polynomials can be put into simplified form by the use of symbolic notation. Replacing $=$ by $\doteq$ implies that exponents will be lowered to subscripts on any symbol which is undefined here except with subscripts.

For example, the simple Laguerre polynomial $L_n(x)$ satisfies the relation

$$\frac{x^n}{n!} = \sum_{k=o}^{n} \frac{(-1)^k n! L_k(x)}{k!(n-k)!} \tag{4.1}$$

In symbolic notation, since L without a subscript is not defined here, we may write equation (4.1) as

$$\frac{x^n}{n!} \doteq \{1-L(x)\}^n$$

Thus more generally if

$$f(x) = \sum_{k=o}^{\infty} c_k x^k$$

and $\{g_k(x)\}$ is a given sequence of functions, then

$$f(g) = \sum_{k=o}^{\infty} c_k g_k(x) .$$

In this chapter we shall give some sufficient conditions so that the polynomial sets defined by means of relations like $\phi_n(\psi(x))$ will satisfy Turán like inequalities.

<u>Theorem 14.</u> <u>The sequence of polynomials</u> $\{H_n(xZ(y))\}$ <u>will satisfy the Turán inequality if</u>

$$\sum_{n=o}^{\infty} Z_n(y) \frac{(2xt)^n}{n!} = e^{-\alpha t^2+\beta t} \Pi (1+\beta_n t) e^{-\beta_n t}$$

<u>where</u> α, β <u>and</u> β_n <u>are functions of</u> x <u>and</u> y ; $\alpha \geq 0$, <u>and</u> β_n <u>are real and</u> $\sum \beta_n^2$ <u>is convergent</u>.

Proof: Consider

$$\sum_{n=o}^{\infty} \frac{H_n(xZ(y))}{n!} t^n \doteq \sum_{n=o}^{\infty} \frac{t^n}{n!} \sum_{k=o}^{[n/2]} \frac{(-1)^k n! (2x)^{n-2k} Z_{n-2k}(y)}{k!(n-2k)!}$$

But

$$\sum_{n=o}^{\infty} \sum_{k=o}^{[n/2]} B(k,n) = \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} B(k,n+2k)$$

Therefore, the right hand side becomes

$$= \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} \frac{t^{n+2k}(-1)^k (2x)^n Z_n(y)}{k!n!}$$

$$= \sum_{n=o}^{\infty} (2xt)^n \frac{Z_n(y)}{n!} \sum_{k=o}^{\infty} \frac{(-t^2)^k}{k!}$$

$$= e^{-t^2} \sum_{n=o}^{\infty} \frac{Z_n(y)}{n!} (2tx)^n$$

$$= e^{-t^2} e^{-\alpha t^2 + \beta t} \Pi(1+\beta_n t) e^{-\beta_n t}$$

$$= e^{-(\alpha+1)t^2 + \beta t} \Pi(1+\beta_n t) e^{-\beta_n t}$$

Thus by Theorem 3, $H_n(xZ(y))$ satisfies the Turán inequality.

For example: Let $Z_n(y) = P_n(y)$, where $P_n(y)$ is the nth degree Legendre polynomial. It is easy to see that

$$\sum_{n=o}^{\infty} \frac{H_n(xP(y))}{n!} t^n = e^{-t^2} \sum_{n=o}^{\infty} \frac{P_n(y)}{n!} (2tx)^n$$

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But

$$\sum_{n=o}^{\infty} P_n(y) \frac{(2tx)^n}{n!} = e^{2txy} J_o((1-y^2)^{\frac{1}{2}} 2tx)$$

$$\therefore \qquad \sum_{n=o}^{\infty} \frac{H_n(xP(y))}{n!} t^n = e^{2txy-t^2} J_o((1-y^2)^{\frac{1}{2}} 2tx)$$

where $J_o(x)$ is the Bessel function of order zero. We know, by Lommel's theorem on the reality of the zeros of $J_\nu(z)$, that if the order ν exceeds -1, then the function $J_\nu(z)$ has no zeros which are not real. Moreover, it is possible to express $J_\nu(z)$ as a product of 'simple factors' of Weierstrassian type, each factor vanishing at one of the zeros of $J_\nu(z)$. The zeros of $z^{-\nu} J_\nu(z)$ are taken to be $\pm j_{\nu,1}, \pm j_{\nu,2}, \pm j_{\nu,3}, \ldots$ and all are real, where $\nu > -1$ and its value in the product form is as below

$$z^{-\nu} J_\nu(z) = \frac{(\frac{1}{2})^\nu}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} \{(1- \frac{z}{j_{\nu,n}}) e^{z/j_{\nu,n}}\} \prod_{n=1}^{\infty} \{(1+ \frac{z}{J_{\nu,n}}) e^{-z/j_{\nu,n}}\}$$

or the formula may also be written in the modified form

$$z^{-\nu} J_\nu(z) = \frac{(\frac{1}{2})^\nu}{\Gamma(\nu+1)} \prod_{n=1}^{\infty} (1- \frac{z^2}{j_{\nu,n}^2})$$

where $\nu > -1$.

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Thus by theorem 3 , the assertion follows.

In particular, the theorem is true for the following cases:

(i) $Z_n(y) = H_n(y)$

(ii) $Z_n(y) = \dfrac{L_n^{(\alpha)}(y)}{L_n^{(\alpha)}(0)}$, $\alpha > -1$

(iii) $Z_n(y) = \dfrac{P_n^{(\lambda)}(y)}{P_n^{(\lambda)}(1)}$, $\lambda > -\frac{1}{2}$

where $H_n(y)$, $L_n^{(\alpha)}(y)$ and $P_n^{(\lambda)}(y)$ have the usual meanings.

<u>Corollary</u> The polynomial set $\{H_n(Z(y))\}$ satisfies the Turán inequality if

$$\sum_{n=o}^{\infty} Z_n(y) \frac{(2t)^n}{n!} = e^{-\alpha t^2+\beta t}\Pi(1+\beta_n t)e^{-\beta_n t}$$

where $\alpha \geq 0$, β and β_n are real and $\sum\beta_n^2$ is convergent.

<u>Example 1</u>. If $Z_n(y) = H_n(y)$, we know that $H_n(H(y))$ is again a the Hermite polynomial, hence Turán inequality is satisfied.

<u>Example 2</u>. If $Z_n(y) = P_n(y)$, where $P_n(y)$ is the Legendre polynomial

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of the nth degree, then

$$\sum_{n=o}^{\infty} \frac{H_n(P(y))}{n!} t^n = e^{-t^2} \sum_{n=o}^{\infty} \frac{P_n(y)}{n!} (2t)^n$$

$$= e^{-t^2} . e^{2yt} J_o((1-y^2)^{\frac{1}{2}} . 2t)$$

$$= e^{2yt-t^2} J_o((1-y^2)^{\frac{1}{2}} . 2t)$$

Thus, by an example on page 72, the result follows.

Example 3. Exactly on the same lines, the result follows for

$$Z_n(y) = \frac{L_n^{(\alpha)}(y)}{L_n^{(\alpha)}(0)}$$

or

$$Z_n(y) = \frac{P_n^{(\lambda)}(y)}{P_n^{(\lambda)}(1)}$$

where the symbols have their usual meanings.

2. Another notation. We introduce the notation $\phi_n(!y(x))$, where ϕ_n is a polynomial of the nth degree. Replacing $=$ by $\doteqdot$ implies

that the exponent will be lowered to the subscript on any symbol which is undefined here except with subscripts, and multiplied by the factorial of the exponent.

For example: $[!y(x)]^k \doteq k!y_k(x)$

Theorem 15. Let $\phi_n(x,y) = L_n(!xZ(y))$, then $\{\phi_n(x,y)\}$ satisfies the Turán inequality if

$$\sum_{k=o}^{\infty} Z_k(y)\,\frac{(-xt)^k}{k!} = e^{-\alpha t^2+\beta t}\Pi(1+\beta_n t)e^{-\beta_n t}$$

where $\alpha \geq 0$, β and β_n are real and $\sum\beta_n^2$ is convergent and α, β and β_n are functions of x and y.

Proof:

$$\sum_{n=o}^{\infty} \phi_n(x,y)\,\frac{t^n}{n!} = \sum_{n=o}^{\infty} \frac{t^n}{n!} \sum_{k=o}^{n} \frac{(-1)^k n!k!x^k Z_k(y)}{(k!)^2(n-k)!}$$

$$= \sum_{n=o}^{\infty} \frac{t^n}{n!} \sum_{k=o}^{\infty} \frac{(-1)^k t^k x^k Z_k(y)}{k!}$$

$$= e^t \sum_{k=o}^{\infty} Z_k(y)\,\frac{(-xt)^k}{k!}$$

Since

$$\sum_{k=0}^{\infty} Z_k(y) \frac{(-xt)^k}{k!} = e^{-\alpha t^2+\beta t} \Pi (1+\beta_n t) e^{-\beta_n t}$$

$$\therefore \qquad \sum_{n=0}^{\infty} \phi_n(x,y) \frac{t^n}{n!} = e^{-\alpha t^2+(\beta+1)t} \Pi (1+\beta_n t) e^{-\beta_n t}$$

Hence by theorem 3 , $\{\phi_n(x,y)\}$ satisfies the Turán inequality.

In particular $L_n(!Z(y))$ also satisfies the Turán inequality.

<u>For example</u>: $\sum_{n=0}^{\infty} \frac{L_n(!y(x))}{n!} t^n = e^t \sum_{k=0}^{\infty} y_k(x) \frac{(-t)^k}{k!}$

Let $y_k(x) = H_k(x)$ then

$$\sum_{n=0}^{\infty} \frac{L_n(!H(x))}{n!} t^n = e^t . e^{-2xt-t^2} = e^{(1-2x)t-t^2}$$

Hence $\{L_n(!H(x))\}$ satisfies the Turán inequality by theorem 3.

3. Miscellaneous results

<u>Theorem 16.</u> <u>Let</u> $\alpha > 0$, $\beta > 0$ <u>and</u>

$$Q_n(\alpha,\beta;x) = \frac{P_n(\alpha,\beta;x)}{h_n} ,$$

where

$$h_n = \frac{(1+\alpha+\beta)_{2n}}{(1+\alpha+\beta)_n n! 2^n} ,$$

is the coefficient of x^n in $P_n(\alpha,\beta;x)$ and $P_n(\alpha,\beta;x)$ is the Jacobi polynomial of the nth degree, then

$$Q_n^2(\alpha,\beta;x) - Q_{n-1}(\alpha+1,\beta+1;x)Q_{n+1}(\alpha-1,\beta-1;x) \geq 0 .$$

Proof: We know

$$P_n(\alpha,\beta;x) = \frac{(1+\alpha+\beta)_{2n}(1+x)^n}{(1+\alpha+\beta)_n n! 2^n} \, {}_2F_1(-n,-\beta-n;-\alpha-\beta-2n; \frac{2}{1+x})$$

$$\therefore \qquad Q_n(\alpha,\beta;x) = (1+x)^n {}_2F_1(-n,-\beta-n;-\alpha-\beta-2n; \frac{2}{1+x})$$

We are going to apply Theorem 4 for the above result. Consider

$$\sum_{n=o}^{N} \binom{N}{n} \frac{z^n}{(1+x)^n} Q_n(N-n+\alpha,N-n+\beta;x)$$

$$= \sum_{n=o}^{N} \binom{N}{n} z^n {}_2F_1(-n,-N-\beta;-\alpha-\beta-2N; \frac{2}{1+x})$$

$$= \sum_{n=o}^{N} \binom{N}{n} Z^n \sum_{k=o}^{n} \frac{(-n)_k(-N-\beta)_k}{k!(-\alpha-\beta-2N)_k} \left(\frac{2}{1+x}\right)^k$$

$$= \sum_{k=o}^{N} \frac{(-N-\beta)_k}{k!(-\alpha-\beta-2N)_k} \left(\frac{2}{1+x}\right)^k \sum_{n=k}^{N} \frac{N!}{n!(N-n)!} (-1)^k \frac{n!}{(n-k)!} Z^n$$

$$= \sum_{k=o}^{N} \frac{(-N-\beta)_k}{k!(-\alpha-\beta-2N)_k} (-1)^k \left(\frac{2}{1+x}\right)^k N! \sum_{n=o}^{N-k} \frac{1}{(N-n-k)!} \frac{Z^{n+k}}{n!}$$

$$= \sum_{k=o}^{N} \frac{(-N)_k(-N-\beta)_k}{(-\alpha-\beta-2N)_k k!} \left(\frac{2Z}{1+x}\right)^k \sum_{n=o}^{N-k} \binom{N-k}{n} Z^n$$

$$= \sum_{k=o}^{N} \frac{(-N)_k(-N-\beta)_k}{(-\alpha-\beta-2N)_k k!} \left(\frac{2Z}{1+x}\right)^k (1+Z)^{N-k}$$

$$= (1+Z)^N \sum_{k=o}^{N} \frac{(-N)_k(-N-\beta)_k}{(-\alpha-\beta-2N)_k k!} \left(\frac{2Z}{(1+x)(1+Z)}\right)^k$$

$$= (1+Z)^N {}_2F_1\left(-N,-N-\beta;-\alpha-\beta-2N; \frac{2Z}{(1+x)(1+Z)}\right)$$

$$= (1+y)^N \frac{Z^N}{(1+x)^N} {}_2F_1\left(-N,-N-\beta;-\alpha-\beta-2N; \frac{2}{1+y}\right)$$

where $y = \dfrac{1+x+xZ}{Z}$

$$= \frac{Z^N}{(1+x)^N} Q_N(\alpha,\beta;y)$$

$$= \frac{Z^N}{(1+x)^N} Q_N\left(\alpha,\beta; \frac{1+x+xZ}{Z}\right)$$

Let x be fixed, $-1 < x < 1$. We obtain for the roots of the polynomial in Z, the condition

$$\frac{1+x+xZ}{Z} = x_\nu$$

where x_ν denotes a root of Q_N. Or

$$Z = \frac{1+x}{x_\nu - x}$$

thus the roots are all real. Using the inequality (2.5) we have

$$\left(\frac{Q_{N-1}(\alpha+1,\beta+1;x)}{Q_N(\alpha,\beta;x)}\right)^2 \geq \frac{Q_{N-2}(\alpha+2,\beta+2;x)}{Q_N(\alpha,\beta;x)}$$

or

$$Q_{N-1}^2(\alpha+1,\beta+1;x) \geq Q_{N-2}(\alpha+2,\beta+2;x)Q_N(\alpha,\beta;x)$$

or

$$Q_N^2(\alpha,\beta;x) \geq Q_{N-1}(\alpha+1,\beta+1;x)Q_{N+1}(\alpha-1,\beta-1;x)$$

<u>Theorem 17.</u> <u>The sequence of Jacobi polynomials</u> $\{P_n^{(\alpha,-\alpha)}(x)\}$

satisfies the Turán inequality if

$$0 \le \alpha < 1 \quad \text{and} \quad 0 \le x \le 1$$

or

$$-1 < \alpha \le 0 \quad \text{and} \quad -1 \le x \le 0$$

or α *and* $\alpha + x$ *have the same sign and* $-1 < \alpha < 1$ *and* $-1 \le x \le 1$.

Proof: The Jacobi polynomial, when $\beta = -\alpha$, satisfies the following recurrence relation

$$n(n-1)P_n^{(\alpha,-\alpha)}(x) = (n-1)(2n-1)xP_{n-1}^{(\alpha,-\alpha)}(x) - [(n-1)^2-\alpha^2]P_{n-2}^{(\alpha,-\alpha)}(x)$$

Now

$$\Delta_n(P^{(\alpha,-\alpha)}) = \begin{vmatrix} P_n^{(\alpha,-\alpha)}(x) & P_{n+1}^{(\alpha,-\alpha)}(x) \\ P_{n-1}^{(\alpha,-\alpha)}(x) & P_n^{(\alpha,-\alpha)}(x) \end{vmatrix}$$

$$= \frac{1}{n(n-1)(n+1)} \begin{vmatrix} n(n-1)P_n^{(\alpha,-\alpha)}(x) & n(n+1)P_{n+1}^{(\alpha,-\alpha)}(x) \\ (n-1)P_{n-1}^{(\alpha,-\alpha)}(x) & (n+1)P_n^{(\alpha,-\alpha)}(x) \end{vmatrix}$$

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$$= \frac{1}{n(n-1)(n+1)} \begin{vmatrix} (n-1)(2n-1)xP_{n-1}^{(\alpha,-\alpha)}(x) & n(2n+1)xP_{n}^{(\alpha,-\alpha)}(x) \\ -[(n-1)^2-\alpha^2]P_{n-2}^{(\alpha,-\alpha)}(x) & -[n^2-\alpha^2]P_{n-1}^{(\alpha,-\alpha)}(x) \\ (n-1)P_{n-1}^{(\alpha,-\alpha)}(x) & (n+1)P_{n}^{(\alpha,-\alpha)}(x) \end{vmatrix}$$

$$= \frac{1}{n(n-1)(n+1)} \begin{vmatrix} -[(n-1)^2-\alpha^2]P_{n-2}^{(\alpha,-\alpha)}(x) & xP_{n}^{(\alpha,-\alpha)}(x) - [n^2-\alpha^2]P_{n-1}^{(\alpha,-\alpha)}(x) \\ (n-1)P_{n-1}^{(\alpha,-\alpha)}(x) & (n+1)P_{n}^{(\alpha,-\alpha)}(x) \end{vmatrix}$$

or

$$n(n-1)(n+1)\Delta_n(P^{(\alpha,-\alpha)}) = -[(n-1)^2-\alpha^2](n+1)P_n^{(\alpha,-\alpha)}(x)P_{n-2}^{(\alpha,-\alpha)}(x)$$

$$- x(n-1)P_n^{(\alpha,-\alpha)}(x)P_{n-1}^{(\alpha,-\alpha)} + (n-1)[n^2-\alpha^2]P_{n-1}^{2(\alpha,-\alpha)}(x)$$

$$= (n-1)(n^2-\alpha^2)\Delta_{n-1}(P^{(\alpha,-\alpha)}) + (n+2\alpha^2-1)P_n^{(\alpha,-\alpha)}(x)P_{n-2}^{(\alpha,-\alpha)}(x)$$

$$+ P_n^{(\alpha,-\alpha)}(x)[n(n-1)P_n^{(\alpha,-\alpha)}(x) - 2n(n-1)xP_{n-1}^{(\alpha,-\alpha)}(x)$$

$$+ \{(n-1)^2-\alpha^2\}P_{n-2}^{(\alpha,-\alpha)}(x)]$$

$$= (n-1)(n^2-\alpha^2)\Delta_{n-1}(P^{(\alpha,-\alpha)}) + n(n-1)P_n^{(\alpha,-\alpha)^2}(x)$$

$$+ [n^2-2n+1-\alpha^2+n+2\alpha^2-1]P_n^{(\alpha,-\alpha)}(x)P_{n-2}^{(\alpha,-\alpha)}(x)$$

$$- 2n(n-1)xP_n^{(\alpha,-\alpha)}(x)P_{n-1}^{(\alpha,-\alpha)}(x)$$

$$= [n^2(n-2)+n(1-\alpha^2)]\Delta_{n-1}(P^{(\alpha,-\alpha)}) + (n^2-n+\alpha^2)P_{n-1}^{(\alpha,-\alpha)^2}(x)$$

$$+ n(n-1)P_n^{(\alpha,-\alpha)^2}(x) - 2n(n-1)xP_n^{(\alpha,-\alpha)}(x)P_{n-1}^{(\alpha,-\alpha)}(x) .$$

or

$$n(n-1)(n+1)\Delta_n(P^{(\alpha,-\alpha)}) = [n^2(n-2)+n(1-\alpha^2)]\Delta_{n-1}(P^{(\alpha,-\alpha)}) + \alpha^2P_{n-1}^{(\alpha,-\alpha)^2}(x)$$

$$+ n(n-1)(1-x^2)P_n^{(\alpha,-\alpha)^2}(x) + (n^2-n)[xP_n^{(\alpha,-\alpha)}(x)-P_{n-1}^{(\alpha,-\alpha)}(x)]^2$$

for $n \geq 2$

Now we consider the value of

$$\Delta_1(P^{(\alpha,-\alpha)})$$

We know

$$P_1^{(\alpha,-\alpha)}(x) = (x+\alpha)$$

$$2P_2^{(\alpha,-\alpha)}(x) = 3xP_1^{(\alpha,-\alpha)}(x) - (1-\alpha^2)$$

or

$$P_2^{(\alpha,-\alpha)}(x) = \frac{3x}{2} P_1^{(\alpha,-\alpha)}(x) - \frac{(1-\alpha^2)}{2}$$

$$\Delta_1(P^{(\alpha,-\alpha)}) = (x+\alpha)^2 - \frac{3x}{2}(x+\alpha) + \frac{(1-\alpha^2)}{2}$$

$$= x^2 + \alpha^2 + 2\alpha x - \frac{3x^2}{2} - \frac{3\alpha x}{2} + \frac{1}{2} - \frac{\alpha^2}{2}$$

$$= -\frac{x^2}{2} + \frac{\alpha^2}{2} + \frac{\alpha x}{2} + \frac{1}{2}$$

$$= \frac{1}{2}(1-x^2) + \frac{\alpha}{2}(\alpha+x) .$$

It is positive if $0 \leq \alpha < 1 \quad 0 \leq x \leq 1$

or $-1 < \alpha \leq 0 \quad -1 \leq x \leq 0$.

Thus $\Delta_n(P^{(\alpha,-\alpha)})$ is positive if $0 \leq \alpha < 1$ and $0 \leq x \leq 1$

or $-1 < \alpha \leq 0$ and $-1 \leq x \leq 0$.

Also $\Delta_n(P^{(\alpha,-\alpha)}$ is positive if α and $(\alpha+x)$ have the same sign.

4. Lakshman Rao [39], [40], [41] has studied the nature of relative maxima and relative minima of the Turán expressions. He has

considered the relative maxima and minima of $e^{-x^2}\Delta_n(H)$, $\Delta_n(J)$ and $e^{-x} \cdot x^{\alpha-1}\Delta_n(L^{(\alpha)})$, where $\Delta_n(H)$, $\Delta_n(J)$ and $\Delta_n(L^{(\alpha)})$ are the Turán expressions for the Hermite polynomials, the Bessel functions and the general Laguerre polynomials respectively.

Let us denote the relative maxima and minima at several points by $M_{1,n}, M_{2,n}, \ldots$ and $m_{1,n}, m_{2,n}, \ldots$ respectively.

The following results have been proved by Rao.

1. The relative maxima of $e^{-x^2}\Delta_n(H)$ occur at the zeros of $H_n(x)$ and the relative minima occur at the zeros of $H_{n-1}(x)$.

2. The successive relative maxima and minima of $e^{-x^2}\Delta_n(H)$ each form an increasing sequence as x decreases from ∞ to 0 . In other words

$$M_{1,n} < M_{2,n} < \ldots M_{r,n} < \ldots$$

$$m_{1,n} < m_{2,n} < \ldots m_{r,n} < \ldots$$

3. Lakshman Rao proved that $(4n-1)M_{r,n} < (4n+1)m_{r,n}$, where $M_{r,n}$ and $m_{r,n}$ are the relative maxima and minima of $e^{-x^2}\Delta_n(H)$.

4. For a fixed r $([n] \geq r \geq 1)$, the relative maxima $M_{r,n}$ form a sequence of increasing functions of n and the relative minima $m_{r,n}$ form a sequence of increasing functions of n , where $M_{r,n}$ and

$m_{r,n}$ are the relative maxima and minima of $e^{-x^2}\Delta_n(H)$ respectively.

5. When $n > 0$, the relative maxima of $\Delta_n(J)$ occur at the zeros of $J_{n-1}(x)$ and the relative minima at the zeros of $J_{n+1}(x)$.

6. The sequences of relative maxima and minima of $\Delta_n(J)$ are decreasing beyond a certain value of r. More specifically

$$M_{r,n} > M_{r+1,n} \quad \text{if} \quad x_{r,n-1} > \xi = \sqrt{2n(n-2)}$$

$$m_{r,n} > m_{r+1,n} \quad \text{if} \quad x_{r,n+1} > \eta = \sqrt{2n(n+2)} \quad .$$

7. The rth relative maximum of $\Delta_n(J)$ is greater than the rth relative minimum, i.e. $M_{r,n} > m_{r,n}$.

8. For a fixed value of r, the rth relative maxima of $\Delta_n(J)$ form a sequence of decreasing functions of n i.e. $M_{r,n} > M_{r,n+1}$ and for a fixed value of r, the rth relative minima of $\Delta_n(J)$ form a sequence of decreasing functions of n i.e. $m_{r,n} > m_{r,n+1}$.

9. The relative maxima and minima of the function $T_n(x) = e^{-x}.\ x^{\alpha-1}\Delta_n(L^{(\alpha)})$ occur at the zeros of $L_{n+1}^{(\alpha)}(x) - 2L_n^{(\alpha)}(x) + L_{n-1}^{(\alpha)}(x)$ and relative minima occur at the zeros of $L_n^{(\alpha)}(x)$.

Let

$$0 < \beta_1 < \beta_2 \ldots$$

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and

$$0 < \delta_1 < \delta_2 \cdots$$

denote the zeros of $L_n^{(\alpha)}(x)$ and $L_{n+1}^{(\alpha)}(x) - 2L_n^{(\alpha)}(x) + L_{n-1}^{(\alpha)}(x)$.
Let

$$M_{r,n} = T_n(\delta_r) \quad \text{and} \quad m_{r,n} = T_n(\beta_r)$$

denote the rth relative maxima and minima respectively of $T_n(x)$.

10. If $\alpha > 3$, the sequence $\{M_{r,n}\}$ is increasing when $\delta_r < \xi$ and decreasing when $\delta_r > \xi$. If $\alpha > 1$, the sequence $\{m_{r,n}\}$ is increasing when $\beta_r < \eta$ and is decreasing when $\beta_r > \eta$, where

$$\xi = \frac{(\alpha-1)(\alpha-3)}{2n+\alpha+1} \quad \text{and} \quad \eta = \frac{\alpha^2-1}{2n+\alpha+1} \quad .$$

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